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مؤتة للبحوث والدراسات

مجلة علمية محكمة ومفهرسة

سلسلة العلوم الطبيعية والتطبيقية

تصدر في جامعة مؤتة

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بسم الله الرحمن الرحيم

مؤتة للبحوث والدراسات

"سلسلة العلوم الطبيعية والتطبيقية"

مجلة علمية محكّمة مفهرسة

تصدر عن جامعة مؤتة

تصدر المجلة مجلداً سنوياً .

شروط النشر:

1- تنشر المجلة البحوث العلمية الأصيلة التي تتوافر فيها شروط البحث في التحديد والإحاطة والاستقصاء والتوثيق في العلوم الطبيعية والتطبيقية للباحثين من داخل جامعة مؤتة وخارجها، مكتوبة باللغة العربية أو الإنجليزية، ويشترط في البحث ألا يكون قد نُشِرَ أو قدِّم للنشر في أيّ مكانٍ آخر، وأن يوقع الباحث الرئيسي خطياً أنموذج (إقرار وتعهد النشر) المدرج ضمن موقع المجلة الإلكتروني na_darmutah@mutah.edu.jo

2- تخضع البحوث المقدمة للتحكيم المكتوم من أساتذة مختصين حسب الأصول العلميّة المتبعة في المجلة.

تعليمات النّشر:

1- يُطبع البحث باستخدام البرنامج الحاسوبي (MS Word) بمسافات مزدوجة بين الأسطر وهوامش 2,5 سم، وعلى وجه واحد من الورقة (A4)، بحيث لا تزيد عدد صفحاته عن (20) صفحة وفي حدود (6000) كلمة، ونوع الخط وحجمه (Traditional Arabic 14) للبحوث المطبوعة باللغة العربية، و (Times New Roman 14) للبحوث المطبوعة باللغة الإنجليزية، بما في ذلك الأشكال والرسومات والجداول والهوامش والملاحق، وترسل منه أربع نسخ ورقية ونسخة إلكترونية على (CD).

2- أن يستخدم الباحث الأرقام العربية ونظام الوحدات الدولي ومختصرات المصطلحات العلمية المعروفة، شريطة أن تُكتب كاملة أوّل مرة تردُ في النصّ.

3- أن يكتب ملخص للبحث باللغة العربية وآخر بالإنجليزية بما لا يزيد عن (150) كلمة لكل منهما، وعلى ورقتين منفصلتين بحيث يكتب في أعلى الصفحة عنوان البحث واسم الباحث (الباحثين) من ثلاثة مقاطع مع العنوان (البريد والإلكتروني) والرتبة العلمية، وتكتب الكلمات الدالة (Keywords) في أسفل صفحة الملخص بما لا يزيد على خمس كلمات بحيث تعبر عن المحتوى الدقيق للمخطوط.

4- أن تُرقم الأشكال والجداول والرسومات والخرائط على التوالي حسب ورودها في البحث، وتُزود بعناوين، ويُشار إلى كل منها بالتسلسل نفسه في متن البحث، وتقدم بأوراق منفصلة.

5- أن يعتمد الباحث في التوثيق أسلوب هارفارد وذلك على النحو الآتي:

أ. يشار إلى المراجع في متن البحث باسم العائلة للمؤلف الواحد/ الاثنين وسنة النشر. مثال: (هولمز، 1991). (سميث وهوتون، 1997). وإذا زاد عدد المؤلفين عن اثنين فيكتب الاسم الأخير للمؤلف الأول وآخرون، ثم سنة النشر. مثال: (مور وآخرون، 1990). وتكتب الأسماء الكاملة للمؤلفين في قائمة المراجع بغض النظر عن عددهم. وفي حال الإشارة إلى المؤلف في بداية الفقرة فيكتب اسم المؤلف ثم تليه سنة النشر بين قوسين. مثال: هالام (1990)

ب. إذا وردت عدة مراجع للمؤلف نفسه وأنجزت في السنة نفسها فتميز بإضافة حروف هجائية بعد السنة مباشرة (ويلسون، 1994أ. ويلسون، 994 ب).

ج. يمكن استخدام الحواشي من أجل توضيح أي غموض أو شرح مبهم كما في حالة المصطلح الذي يتطلب التوضيح، في هذه الحالة تدرج الحواشي بقائمة بعد انتهاء البحث مباشرة بأرقام متسلسلة حسب ورودها في متن البحث.

د. تدرج المراجع والمصادر المستخدمة في البحث في قائمة واحدة في نهاية البحث (بعد قائمتي الحواشي والملاحق) وترتب هجائياً وفق اسم العائلة للمؤلف وعلى النحو التالي:

الكتب:

هانت، ج. 1996. البترول الجيوكيمياء والجيولوجيا، الطبعة الثانية. جورج فريمان وشركاه، نيويورك.

فصل في كتاب:

شين، أ. ي. 1983. بيئة المد والجزر المستوية، المنشور في: بيئات ترسيب الكربونات، الذي حرره بيبوت، د. ق. ومور، سي. هـ، الجمعية الأمريكية لعلماء جيولوجيا البترول، تولسا، أوكلاهوما، الولايات المتحدة الأمريكية، ص 171-210.

الدوريات:

قندلفت، ف. ب.، ويلسون، أ. ب.، وبيرسون، ب. ن. 2006. البيئة القديمة لعوالق المنخربات من الإيوسين الأوسط المتأخر وتطورها. المستحاثات المجهرية البحرية. 60 (1): 1-16.

المؤتمرات والندوات:

هوبر، ب. ت. 1991. دراسة الطبقات الحيوية للعوالق من مواقع 738 و744، كيرغولن هضبة (جنوب المحيط الهندي). من العصر باليوجين والنيوجين المبكر. والمنشور في: بارون، ج.، لارسن، ب.، وآخرون. (تحرير): وقائع برنامج حفر المحيطات، النتائج العلمية، المجلد، 119. محطة الكلية، ص: 427–449.

الأطروحات:

ثوابته، س. م. 2006. دراسة صخرية ورسوبية وجيوكيميائية الحجر الجيري على طول الجانب الشرقي من وادي الأردن والبحر الميت. أطروحة ماجستير، الجامعة الهاشمية.

تقارير غير منشورة:

مخلوف، ع. م. والحداد، أ.ج. 2006. البيئات الترسيبية والسحنات من تشكيل أبو رويس التابع لأواخر العصر الترياسي، الأردن. مجلة علوم الأرض الآسيوية. انكلترا، في النشر.

لا تعاد المخطوطات المقدمة للنشر في المجلة إلى أصحابها سواءً قُبلت للنشر أم لم تُقبل كما تحتفظ الهيئة بحقها في عدم نشر أي بحث دون إبداء الأسباب، وتُعد قراراتها نهائية.

تحتفظ المجلة بحقها في أن تختزل أو تُعيد صياغة بعض الجمل لأغراض الضبط اللغوي ومنهج التحرير.

يُهدى إلى الباحث (الباحثين) نسخة واحدة من العدد المنشور فيه البحث و(20) مستلة منه، ويتحمل الباحث (الباحثون) نفقات أي مستلات إضافية.

تتم المراسلات جميعها باسم:

الأستاذ الدكتور أحمد خلف السكارنه

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مجلة علمية محكّمة ومفهرسة تصدر عن

عمادة البحث العلمي

جامعة مؤتة

قسيمة اشتراك

أرجو قبول اشتراكي في مجلة مؤتة للبحوث والدراسات:

☐ سلسلة العلوم الإنسانية والاجتماعية ☐ سلسلة العلوم الطبيعية والتطبيقية

للمجلد رقم ( ) الاسم :........................ العنوان : ....

التاريخ : / / 200 التوقيع :............ ....

طريقة الدفع : ☐ شيك ☐ حوّالة بنكية ☐ حوّالة بريدية

أ – داخل الأردن : للأفراد (9) دنانير أردنية.

للمؤسسات (11) ديناراً أردنياً.

ب – خارج الأردن (للأفراد والمؤسسات): (30) دولاراً أمريكياً.

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تُملأ هذه القسيمة، وترسل مع قيمة الاشتراك إلى العنوان التالي:

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المحتويات

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Generalization of A Retard Gronwall-Like Inequality

M. H. M. Rashid *

Kamal Al-Dawoud

Abstract

This paper generalizes a Pachpatte result of Gronwall-like inequalities [Pachpatte, 2002] to a new type of retarded inequalities which includes both a nonconstant term outside theintegrals and more than one distinct nonlinear integrals. From our result Bihari's result and Pachpatte result can be deduced as some special cases. Our result can be applied to give the global existence and estimate solutions.

Keywords: Explicit bounds, Integral inequality, Gronwall inequality.

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تعميـــم تعـــــوق متباينـــــــة غرونـــــــوال

محمد حسين محمد رشيد

كمال الداود

ملخص

في هذا البحث تم تعميم متباينة غرونوال ونتيجة (Pachptte) الى نوع جديد من المتباينات المتأخرة والتي تشمل حدود غير ثابتة خارج التكامل وأكثر من التكاملات غير الخطية المتميزة. من خلال نتائجنا يمكن الحصول على نتائج غرونوال و(Pachptte) كحالات خاصة. ويمكن تطبيق نتيجة لدينا لإعطاء وجود وتقدير الحلول شاملة.

1. Introduction

Gronwall inequality is an important tool in the study of existence, uniqueness, boundedness, stability, invariant manifolds and other qualitative properties of solutions of differential equations and integral equations. Many results on its generalization can be found for example in [Agarwaland Thandapani (1981),Bellman (1943a), Bellman (1943b), Bihari (1964), Bihari (1965), Oguntuase (2000), Oguntuase (2001), Pachpatte (1965), Pachpatte (1995), Pachpatte (2002), Rashid (2012)]. Among them one of the important things is Bihari's [Bihari, 1965] for the nonlinear inequality

(1.1) $$u(t) \leq c + \int_0^t f(s)u(s)ds, t \geq 0$$

where $c > 0$ is a constant. replacing c by a function $g(t)$ in (1.1). Lipovan [8]investigatesthe retarded Gronwall-like inequalities

(1.2) $$u(t) \leq c + \int_{\alpha(a)}^{\alpha(t)} g(s)K\big(u(s)\big)ds, a \leq t \leq b,$$

And

(1.3) $$u(t) \leq c + \int_{\alpha(a)}^{\alpha(t)} g(s)K\big(u(s)\big)ds + \int_{\alpha(a)}^{\alpha(t)} h(s)K\big(u(s)\big)ds, a \leq t \leq b,$$

However, sometimes we need to study such inequalities with a function $f(t)$ in place of the constant term c. For example, in order to prove almost periodicity ofan invariant manifold [Zhang, 1993] we consider an integral inequality with such a functionterm $f(t)$.

In this paper we consider such an inequality

(1.4) $$u^p(t) \leq f(t) + \int_a^t g(s)\psi_1\big(u(s)\big)ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2\big(u(s)\big)ds, \quad p \geq 1$$

where $f(t)$ is a function and ψ_i, $i = 1,2$may be distinct,and improve Pachpatteresult [Pachpatte, 2002]. Furthermore, it easily seen that the result of [Agarwal, R.P., Deng, S. and Zhang (2005), Pachpatte (2002)] can be deducedfrom our result as some special cases.

2. main results

In what follows, $\mathbb{R}$denotes the set of real numbers; $\mathbb{R}_+ = [0, \infty[$, $R_+^* =]0, \infty[, \mathbb{R}_1 = [1, \infty[$

is the subset of $\mathbb{R}$; and ' denotes the derivative. $C(J, \mathbb{R})$denotes the set of all continuous functions from J into $\mathbb{R}_+$ and $C^1(J, J)$ denotesthe set of all continuously differentiable functions from J into J.

Theorem 2.1. Let $u, g, h \epsilon C(J, \mathbb{R}_+)$, $\alpha, \beta \epsilon C^1(J, J)$ be non-decreasing with $\alpha(t) \leq t \leq \beta(t)$ on J,$u(\beta(t)) = u(\alpha(t)) = u(t), \alpha(a) = \beta(a)$ and $k > 0$ is a constant. If the inequality

(2.1)$u(t) \leq k + \int_a^t g(s) u(s) ds + \int_{\alpha(t)}^{\beta(t)} h(s) u(s) ds$

for $t \in J$, then

(2.2)$u(t) \leq k \times exp[G(t) + H(t)],$

where

(2.2a)$G(t) = \int_a^t g(s) ds,$

(2.2b)$H(t) = \int_{\alpha(t)}^{\beta(t)} h(s) ds.$

Proof. Define a function $v(t)$ to be

(2.3) $v(t) = k + \int_a^t g(s) u(s) ds + \int_{\alpha(t)}^{\beta(t)} h(s) u(s) ds.$

Then

(2.4) $u(t) \leq v(t)$ and $v(a) = k.$

Differentiate (2.3) , we have

(2.5) $v'(t) = g(t) u(t) + h(\beta(t)) \beta'(t) u(\beta(t)) - h(\alpha(t)) \alpha'(t) u(\alpha(t))$

$= g(t) u(t) + [h(\beta(t)) \beta'(t) - h(\alpha(t)) \alpha'(t)] u(t)$

$\leq g(t)v(t) + [h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]v(t).$

Therefore

(2.6) $$\frac{v'(t)}{v(t)} \leq g(t) + [h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)].$$

Integrating (2.6) from a to t, and making change of variable we have

(2.7) $$v(t) \leq k \times \exp[G(t) + H(t)],$$

where $G(t)$ and $H(t)$ are defined by (2.2a) and (2.2b). Hence by (2.4) we have the desired result.

Remark: With $\alpha(t) = \beta(t)$ in the Theorem we obtain the celebrated GronwallBellman inequality [Lipovan, 2000].

Corollary 2.2. Let $,g,h \in C(J,\mathbb{R}_+)$, $f \in C(J,\mathbb{R}_+^*)$ and $\alpha,\beta \in C^1(J,J)$be non-decreasingwith $\alpha(t) \leq t \leq \beta(t)$ on J,$u(\beta(t)) = u(\alpha(t)) = u(t), \alpha(a) = \beta(a)$. If the inequality

(2.8) $u(t) \leq f(t) + \int_a^t g(s)u(s)ds + \int_{\alpha(t)}^{\beta(t)} h(s)u(s)ds$

for $t \in J$, then

(2.8a) $u(t) \leq f(t) \times exp[G(t) + H(t)],$

where

(2.8b) $G(t) = \int_a^t g(s)ds,$

(2.8c) $H(t) = \int_{\alpha(t)}^{\beta(t)} h(s)ds.$

Proof. Since $f(t)$ is positive and non-decreasing we can restate (2.8) as

(2.9) $\frac{u(t)}{f(t)} \leq 1 + \int_a^t \frac{g(s)}{f(s)}u(s)\,ds + \int_{\alpha(t)}^{\beta(t)} \frac{h(s)}{f(s)}u(s)\,ds$

Let $r(t) = \dfrac{u(t)}{f(t)}$, then we have

(2.10) $r(t) \le 1 + \int_a^t r(s)u(s)ds + \int_{\alpha(t)}^{\beta(t)} r(s)u(s)ds$

Then it follows from Theorem 2.1 that

(2.11) $r(t) \le \exp[G(t) + H(t)]$

where $G(t)$ and $H(t)$ are defined by (2.8b) and (2.8c). Therefore

(2.12) $u(t) \le f(t) \times exp[G(t) + H(t)]$.

Theorem 2.3. Let $u, g, h \in C(J, \mathbb{R}_+)$, $g, h \in C(J, \mathbb{R}_+^*)$ and $\alpha, \beta \in C^1(J, J)$ be non-decreasing with $\alpha(t) \le t \le \beta(t)$ on J, $u(\beta(t)) = u(\alpha(t)) = u(t), \alpha(a) = \beta(a)$ and $k > 0$ is a constant . If the inequality

(2.13) $u(t) \le k + \int_a^t g(s)u(s)\ln u(s)\, ds + \int_{\alpha(t)}^{\beta(t)} h(s)u(s)\ln u(s)\, ds$

for $t \in J$, then

(2.13a) $u(t) \le k \times \exp[e^{G(t)+H(t)}]$,

where

(2.13b) $G(t) = \int_a^t g(s)ds,$

(2.13c) $H(t) = \int_{\alpha(t)}^{\beta(t)} h(s)ds.$

Proof. Define a function $v(t)$ by the right-hand side of (2.13) we have

(2.14) $v(t) = k + \int_a^t g(s)u(s)\ln u(s)\, ds + \int_{\alpha(t)}^{\beta(t)} h(s)u(s)\ln u(s)\, ds$

Then it is clear that

(2.15) $u(t) \le v(t)$ and $v(a) = 0$.

Differentiate (2.14) we get

$$v'(t) = g(t)u(t)\, lnu(t) + h(\beta(t))\beta'(t)u(\beta(t))\, lnu(\beta(t))$$

(2.16) $-h(\alpha(t))\,\alpha'(t)u(\alpha(t))\,\ln u(\alpha(t))$

$\leq [g(t) + h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]\,u(t)\,\ln u(t)$

$\leq [g(t) + h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]\,v(t)\,\ln v(t)$

Therefore

(2.17) $\frac{v'(t)}{v(t)\ln v(t)} \leq g(t) + h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)$

Integrating (2.17) from a to t and making change of variable, we have

(2.18) $\ln(\ln(v(t)) \leq [H(t) + G(t)\} + C$

where $G(t), H(t)$ are defined by (2.13b) and (2.13c) and C is a constant. Hence

(2.19) $v(t) \leq k \times \exp[e^{H(t)+G(t)}]$.

So by (2.15) we have the desired result.

Theorem 2.4. Let$u, g, h \in C(J, \mathbb{R}_+)$, $g, h \in C(J, \mathbb{R}_+^*)$ and $\alpha, \beta \in C^1(J, J)$ be non-decreasing with $\alpha(t) \leq t \leq \beta(t)$ on J, $u(\beta(t)) = u(\alpha(t)) = u(t), \alpha(a) = \beta(a)$ and $k > 0, p > 1$ are constants. If the inequality

(2.20) $u^p(t) \leq k + \int_a^t g(s)u(s)ds + \int_{\alpha(t)}^{\beta(t)} h(s)u(s)\,ds$

for $t \in J$, then

(2.20a) $u(t) \leq [k^{\frac{p-1}{p}} + (H(t) + G(t)]^{\frac{1}{p-1}}$

where

(2.20b) $G(t) = \int_a^t g(s)ds,$

(2.20c) $H(t) = \int_{\alpha(t)}^{\beta(t)} h(s)ds.$

Proof: Define a function $v(t)$ by the right-hand side of (2.20) we have

(2.21) $v(t) = k + \int_a^t g(s)u(s)ds + \int_{\alpha(t)}^{\beta(t)} h(s)u(s)\,ds$

Then it is clear that

(2.22) $u(t) \leq v^{\frac{1}{p}}(t)$ and $v(a) = k$

Differentiate (2.21), we have

(2.23) $v'(t) = g(t)u(t) + [h(\beta(t))\beta'(t)u(\beta(t)) - h(\alpha(t))\alpha'(t)u(\alpha(t))]$

$= g(t)u(t) + [h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]u(t)$

$\leq g(t)v^{\frac{1}{p}}(t) + [h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]v^{\frac{1}{p}}(t)$

Hence

(2.24) $v^{-\frac{1}{p}}(t)v'(t) \leq g(t) + [h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]$

Integrating from a to t and making change of variable, we get

(2.25) $v(t) \leq [k^{\frac{p-1}{p}} + (H(t) + G(t)]^{\frac{1}{p-1}}$.

Therefore

(2.26) $u(t) \leq [k^{\frac{p-1}{p}} + (H(t) + G(t)]^{\frac{1}{p-1}}$.

Theorem 2.5. Let $u, g, h \in C(J, \mathbb{R}_+)$, and $\alpha, \beta \in C^1(J, J)$ be non-decreasing with $\alpha(t) \leq t \leq \beta(t)$ on J, $u(\beta(t)) = u(\alpha(t)) = u(t), \alpha(a) = \beta(a)$ and $k > 0, p > 1$ are constants. If the inequality

(2.27) $u(t) \leq k + \int_a^t g(s)u(s)ds + \int_{\alpha(t)}^{\beta(t)} h(s)u^p(s)ds$

for $t \in J$, then

(2.27a) $u(t) \leq [w(t)]^{\frac{1}{p-1}}$,

where

(2.27b) $w(t) \leq \exp[(1-p)G(t)] \times$

$$\int_a^t \{\exp[(p-1)G(\mu)](1-p)[h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]\}d\mu + k.$$

Proof. Define a function $v(t)$ by the right-hand side of (2.27) we have

(2.28) $v(t) = k + \int_a^t g(s)u(s)ds + \int_{\alpha(t)}^{\beta(t)} h(s)u^p(s)ds$

Then it is clear that

(2.29) $u(t) \leq v(t)$ and $v(a) = k$.

Differentiate (2.28), we have

$v'(t) = g(t)u(t) + h(\beta(t))\beta'(t)u^p(\beta(t)) - h(\alpha(t))\alpha'(t)u^p(\alpha(t))$

$= g(t)u(t) + [h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]u^p(t)$

(2.30) $\leq g(t)v(t) + [h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]v^p(t)$.

Let $w(t) = v^{1-p}(t)$, then

(2.31) $v'(t) = \frac{1}{1-p} w'(t)v^p(t)$

and

(2.32) $v(t) = v^p(t)w(t)$.

Substitute (2.31) and (2.32) into (2.30) we have

(2.33) $\frac{1}{1-p} w'(t)v^p(t) - g(t)v^p(t)w(t) \leq [h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]v^p(t)$.

Hence

(2.34) $w'(t) - (1-p)g(t)w(t) \leq (1-p)[h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]$.

Then

(2.35) $[\exp[(p-1)G(t)w(t)]' \leq \exp[(p-1)G(t) \times$

$(1-p)[h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]$

where

$$G(t) = \int_a^t g(s)\,ds.$$

Integrating from a to t, we get

(2.36) $w(t) \leq \exp[(1-p)G(t)] \times$

$$\int_a^t \{\exp[(p-1)G(\mu)]\,(1-p)\left[h(\beta(t))\beta'(t)-h(\alpha(t))\alpha'(t)\right]\}d\mu+k.$$

So

(2.37) $v(t)\le [w(t)]^{\frac{1}{1-p}}.$

Therefore by (2.29) we have the desired result.

Corollary 2.6. Let $u,g,h\in C(J,\mathbb{R}_+)$, $f\in C(J,\mathbb{R}_+^*)$ and $\alpha,\beta\in C^1(J,J)$ be non-decreasing with $\alpha(t)\le t\le\beta(t)$ on J, $u(\beta(t))=u(\alpha(t))=u(t),\alpha(a)=\beta(a)$ and $k>0,p>1$ are constants. If the inequality

(2.38) $u(t)\le f(t)+\int_a^t g(s)u(s)\,ds+\int_{\alpha(t)}^{\beta(t)} h(s)u^p(s)\,ds$

for $t\in J$, then

(2.38a) $u(t)\le f(t)[w(t)]^{\frac{1}{1-p}}$

where $w(t)$ is defined by (2.36).

Proof: Since $f(t)$ is positive and non-decreasing we can restate (2.38) as

(2.39) $\frac{u(t)}{f(t)}\le 1+\int_a^t g(s)\frac{u(s)}{f(s)}ds+\int_{\alpha(t)}^{\beta(t)} h(s)f^{p-1}(s)\frac{u^p(s)}{f^p(s)}ds$.

Let $r(t)=\frac{u(t)}{f(t)}$, then we have

(2.40) $r(t)\le 1+\int_a^t g(s)r(s)ds+\int_{\alpha(t)}^{\beta(t)} h(s)f^{p-1}(s)r^p(s)ds.$

Therefore it follows from Theorem 2.5 that

(2.41) $r(t)\le [w(t)]^{\frac{1}{1-p}}$

where $w(t)$ is defined by (2.36). Hence

(2.42) $u(t)\le f(t)[w(t)]^{\frac{1}{1-p}}.$

Theorem 2.7. Let $u, g, h \in C(\mathbb{R}_+, \mathbb{R}_+)$. Suppose that $\alpha, \beta \in C^1(J, J)$ are non-decreasing with $\alpha(t) \le t \le \beta(t)$ on J, $u(\beta(t)) = u(\alpha(t)) = u(t), \alpha(a) = \beta(a)$. For $i = 1,2$, let $\psi_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a non-decreasing functions with $\psi_i(u) > 0$ for $u > 0$ and $k > 0$ is a constant. If the inequality

(2.43) $u(t) \le k + \int_a^t g(s)\psi_1(u(s))ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2(u(s))ds,$

then for $a \le t \le t_1$,

(i) In case $\psi_1(u) \le \psi_2(u)$,

(2.44) $u(t) \le \Psi_2^{-1}[\Psi_2(k) + G(t) + H(t)]$

(ii) In case $\psi_1(u) \ge \psi_2(u)$,

(2.45) $u(t) \le \Psi_1^{-1}[\Psi_1(k) + G(t) + H(t)]$

where $G(t)$ and $H(t)$ are defined by (2.13b) and (2.13c) and for $i = 1,2$, Ψ_i^{-1}, are the inverse functions of

(2.46) $\Psi_i(\mu) = \int_{\mu_0}^{\mu} \frac{ds}{\psi_i(s)}, \quad \mu > 0, \mu_0 > 0,$

and $t_1 \in J$ is chosen so that

$\Psi_i(k) + G(t) + H(t) \in Dom(\Psi_i^{-1}),$

respectively, for all t lying in the interval $[a, t_1]$.

Proof: Define a function $q(t)$ by the right-hand side of (2.43)

(2.47) $q(t) = k + \int_a^t g(s)\psi_1(u(s))ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2(u(s))ds.$

Then

(2.48) $u(t) \le q(t)$ and $q(a) = k.$

Differentiate (2.47) we get

$q'(t) \le g(t)\psi_1(u(t)) + h(\beta(t))\psi_2(u(\beta(t)))\beta'(t) - h(\alpha(t))\psi_2(u(\alpha(t)))\alpha'(t)$

(2.49) $\le g(t)\psi_1(q(t)) + [h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]\psi_2(q(t))$

Now if $\psi_1(u(t)) \leq \psi_2(u(t))$, then

(2.50) $q'(t) \leq [g(t) + h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]\psi_2(q(t))$.

Hence

(2.51) $\frac{d}{dt}\Psi_2(q(t)) = \frac{q'(t)}{\psi_2(q(t))} \leq [g(t) + h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]$.

Integrating from a to t and making change of variable, we have

(2.52) $\Psi_2(q(t)) \leq (\Psi_2(k) + G(t) + H(t))$.

Therefore

(2.53) $q(t) \leq \Psi_2^{-1}[\Psi_2(k) + G(t) + H(t)]$.

Now by (2.48) we have the desired result. Since the proof of case (ii) is similar we omit the details.

Corollary 2.8. Let $u, g, h, f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $f \in C(J, \mathbb{R}_+^*)$ be non-decreasing. Suppose that $\alpha, \beta \in C^1(J, J)$ are non-decreasing with $\alpha(t) \leq t \leq \beta(t)$ on J, $u(\beta(t)) = u(\alpha(t)) = u(t), \alpha(a) = \beta(a)$. For $i = 1,2$, let $\psi_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a non-decreasing functions with $\psi_i(u) > 0$ for $u > 0, \frac{\psi_i(u(t))}{f(t)} \leq \psi_i\left(\frac{u(t)}{f(t)}\right)$ and $k > 0$ is a constant. If the inequality

(2.43) $u(t) \leq f(t) + \int_a^t g(s)\psi_1(u(s))ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2(u(s))ds$,

then for $a \leq t \leq t_1$,

(i) In case $\psi_1(u) \leq \psi_2(u)$,

(2.44) $u(t) \leq f(t)\Psi_2^{-1}[\Psi_2(1) + G(t) + H(t)]$

(ii) In case $\psi_1(u) \geq \psi_2(u)$,

(2.45) $u(t) \leq f(t)\Psi_1^{-1}[\Psi_1(1) + G(t) + H(t)]$

where $G(t)$ and $H(t)$ are defined by (2.13b) and (2.13c) and for $i = 1,2$, Ψ_i^{-1}, are the inverse functions of

(2.46) $\Psi_i(\mu) = \int_{\mu_0}^{\mu} \frac{ds}{\psi_i(s)}, \quad \mu > 0, \mu_0 > 0,$

and $t_1 \in J$ is chosen so that

$\Psi_i(1) + G(t) + H(t) \in Dom(\Psi_i^{-1})$,

respectively, for all t lying in the interval $[a, t_1]$.

Proof: Since $f(t)$ is positive and non-decreasing we can restate (2.51) as

(2.58) $\frac{u(t)}{f(t)} \leq 1 + \int_a^t g(s)\frac{\psi_1(u(s))}{f(s)}ds + \int_{\alpha(t)}^{\beta(t)} h(s)\frac{\psi_2(u(s))}{f(s)}ds$

$\leq 1 + \int_a^t g(s)\psi_1\left(\frac{u(s)}{f(s)}\right)ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2\left(\frac{u(s)}{f(s)}\right)ds$

Let $r(t) = \frac{u(t)}{f(t)}$, then we have

(2.59) $r(t) \leq 1 + \int_a^t g(s)\psi_1(r(s))ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2(r(s))ds.$

Therefore it follows from Theorem 2.7 (i) that

(2.60) $r(t) \leq \Psi_2^{-1}[\Psi_2(1) + G(t) + H(t)].$

Hence

(2.61) $u(t) \leq f(t)\Psi_2^{-1}[\Psi_2(1) + G(t) + H(t)].$

Theorem 2.9. Let $u, g, h \in C(\mathbb{R}_+, \mathbb{R}_+)$. Suppose that $\alpha, \beta \in C^1(J, J)$ are non-decreasing with $\alpha(t) \leq t \leq \beta(t)$ on J, $u(\beta(t)) = u(\alpha(t)) = u(t), \alpha(a) = \beta(a)$. For $i = 1,2$, let $\psi_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a non-decreasing functions with $\psi_i(u) > 0$ for $u > 0$ and $k > 0$ is a constant. If the inequality

(2.62) $u^p(t) \leq k + \int_a^t g(s)\psi_1(u(s))ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2(u(s))ds,$

then for $a \leq t \leq t_1$,

(i) In case $\psi_1(u) \leq \psi_2(u)$,

(2.44) $u(t) \leq \left(\Psi_2^{-1}[\Psi_2(k) + G(t) + H(t)]\right)^{\frac{1}{p}}$

(ii) In case $\psi_1(u) \geq \psi_2(u)$, (2.45) $u(t) \leq \left(\Psi_1^{-1}[\Psi_1(k) + G(t) + H(t)]\right)^{\frac{1}{p}}$

where $G(t)$ and $H(t)$ are defined by (2.13b) and (2.13c) and for $i = 1,2$, Ψ_i^{-1}, are the inverse functions of

(2.46) $\Psi_i(\mu) = \int_{\mu_0}^{\mu} \frac{ds}{\psi_i\left(s^{\frac{1}{p}}\right)}, \quad \mu > 0, \mu_0 > 0,$

and $t_1 \in J$ is chosen so that

$\Psi_i(k) + G(t) + H(t) \in Dom(\Psi_i^{-1}),$

respectively, for all t lying in the interval $[a, t_1]$.

Proof: Define a function $z(t)$ by the right- hand side of (2.62)

(2.65a) $z(t) = k + \int_a^t g(s)\psi_1(u(s))ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2(u(s))ds.$

Then

(2.66) $u(t) \leq z^{\frac{1}{p}}(t)$ and $z(a) = k.$

Differentiate (2.65a) we get

$z'(t) \leq g(t)\psi_1(u(t)) + h(\beta(t))\psi_2\left(u(\beta(t))\right)\beta'(t) - h(\alpha(t))\psi_2\left(u(\alpha(t))\right)\alpha'(t)$

(2.66a) $\leq g(t)\psi_1\left(z^{\frac{1}{p}}(t)\right) + [h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]\psi_2\left(z^{\frac{1}{p}}(t)\right).$

Now if $\psi_1(u(t)) \leq \psi_2(u(t))$, then

(2.67) $z'(t) \leq [g(t) + h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]\,\psi_2\left(z^{\frac{1}{p}}(t)\right).$

Hence

(2.68) $\frac{z'(t)}{\psi_2\left(z^{\frac{1}{p}}(t)\right)} \leq [g(t) + h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)].$

Integrating from a to t and making change of variable, we have

(2.52) $\Psi_2(z(t)) \leq (\Psi_2(k) + G(t) + H(t)).$

Therefore

(2.53) $z(t) \leq \Psi_2^{-1}[\Psi_2(k) + G(t) + H(t)]$.

Now by (2.66) we have the desired result. Since the proof of case (ii) is similar we omit the details.

Corollary 2.10. Let $u, g, h, f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $f \in C(J, \mathbb{R}_+^*)$ be non-decreasing. Suppose that $\alpha, \beta \in C^1(J, J)$ are non-decreasing with $\alpha(t) \leq t \leq \beta(t)$ on J, $u(\beta(t)) = u(\alpha(t)) = u(t), \alpha(a) = \beta(a)$. For $i = 1,2$, let $\psi_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a non-decreasing functions with $\psi_i(u) > 0$ for $u > 0, \frac{\psi_i(u(t))}{f(t)} \leq \psi_i\left(\frac{u(t)}{f(t)}\right)$ and $k > 0$ is a constant. If the inequality

(2.71) $u^p(t) \leq f^p(t) + \int_a^t g(s)\psi_1(u(s))ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2(u(s))ds,$

then for $a \leq t \leq t_1$,

(i) In case $\psi_1(u) \leq \psi_2(u)$,

(2.72) $u(t) \leq f(t)[\Psi_2^{-1}[\Psi_2(1) + A(t) + B(t)]]^{\frac{1}{p}}$

(ii) In case $\psi_1(u) \geq \psi_2(u)$,

(2.73) $u(t) \leq f(t)(\Psi_1^{-1}[\Psi_1(1) + A(t) + B(t)])^{\frac{1}{p}}$

where

(2.73a) $A(t) = \int_a^t f^{1-p}(s)g(s)\,ds,$

(2.73b) $B(t) = \int_{\alpha(t)}^{\beta(t)} f^{1-p}(s)g(s)\,ds,$

and for $i = 1,2$, Ψ_i^{-1}, are the inverse functions of

(2.74) $\Psi_i(\mu) = \int_{\mu_0}^{\mu} \frac{ds}{\psi_i(s)}, \quad \mu > 0, \mu_0 > 0,$

and $t_1 \in J$ is chosen so that

$\Psi_i(1) + A(t) + B(t) \in Dom(\Psi_i^{-1}),$

respectively, for all t lying in the interval $[a, t_1]$.

Proof: (i) Since $f(t)$ is positive and non-decreasing we can restate (2.72) as

(2.75) $$\frac{u^p(t)}{f^p(t)} \leq 1 + \int_a^t g(s)\frac{\psi_1(u(s))}{f^p(s)}ds + \int_{\alpha(t)}^{\beta(t)} h(s)\frac{\psi_2(u(s))}{f^p(s)}ds$$

$$\leq 1 + \int_a^t g(s)f^{1-p}(s)\psi_1\left(\frac{u(s)}{f(s)}\right)ds + \int_{\alpha(t)}^{\beta(t)} h(s)f^{1-p}(s)\psi_2\left(\frac{u(s)}{f(s)}\right)ds$$

Let $r(t) = \frac{u(t)}{f(t)}$, then we have

(2.76) $$r^p(t) \leq 1 + \int_a^t g(s)f^{1-p}(s)\psi_1(r(s))ds + \int_{\alpha(t)}^{\beta(t)} h(s)f^{1-p}(s)\psi_2(r(s))ds.$$

Therefore it follows from Theorem 2.12 (i) that

(2.77) $$r(t) \leq [\Psi_2^{-1}[\Psi_2(1) + A(t) + B(t)]]^{\frac{1}{p}},$$

where $A(t)$ and $B(t)$ are defined by (2.68a) and (2.68b). Hence

(2.78) $$u(t) \leq f(t)[\Psi_2^{-1}[\Psi_2(1) + A(t) + B(t)]]^{\frac{1}{p}}.$$

Since the proof of case (ii) is similar, we omit the details.

Theorem 2.11. Let $u \in C(J, \mathbb{R}_1)$ and $g, h \in C(\mathbb{R}_+, \mathbb{R}_+)$. Suppose that $\alpha, \beta \in C^1(J, J)$ are non-decreasing with $\alpha(t) \leq t \leq \beta(t)$ on J, $u(\beta(t)) = u(\alpha(t)) = u(t), \alpha(a) = \beta(a)$. For $i = 1,2$, let $\psi_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a non-decreasing functions with $\psi_i(u) > 0$ for $u > 0$ and $k > 0, p > 1$ areconstanst. If the inequality

(2.79) $$u^p(t) \leq k + \int_a^t g(s)\psi_1(\ln(u(s)))ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2(\ln(u(s)))ds,$$

then for $a \leq t \leq t_1$,

(i) In case $\psi_1(u) \leq \psi_2(u)$,

(2.80) $$u(t) \leq \exp[\Psi_2^{-1}[\Psi_2(k) + G(t) + H(t)]]$$

(ii) In case $\psi_1(u) \geq \psi_2(u)$,

(2.81) $u(t) \leq \exp[\Psi_1^{-1}[\Psi_1(k) + G(t) + H(t)]]$

where $G(t)$ and $H(t)$ are defined by (2.13b) and (2.13c) and for $i = 1,2$, Ψ_i^{-1}, are the inverse functions of

(2.82) $\Psi_i(\mu) = \int_{\mu_0}^{\mu} \frac{pe^{ps}ds}{\psi_i(s)}$, $\mu > 0, \mu_0 > 0$,

and $t_1 \in J$ is chosen so that

$\Psi_i(k) + G(t) + H(t) \in Dom(\Psi_i^{-1})$,

respectively, for all t lying in the interval $[a, t_1]$.

Proof: Define a function $z(t)$ by the right- hand side of (2.62)

(2.83) $z(t) = k + \int_a^t g(s)\psi_1(\ln(u(s)))ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2(\ln(u(s)))ds$.

Then

(2.84) $u(t) \leq z^{\frac{1}{p}}(t)$ and $z(a) = k$.

Differentiate (2.83) we get

$z'(t) \leq g(t)\psi_1(\ln(u(t))) + h(\beta(t))\psi_2(\ln(u(\beta(t))))\beta'(t)$

$-h(\alpha(t))\psi_2(\ln(u(\alpha(t))))\alpha'(t)$

(2.85) $\leq g(t)\psi_1\left(\ln(z^{\frac{1}{p}}(t))\right) + [h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]\psi_2\left(\ln(z^{\frac{1}{p}}(t))\right)$.

Now if $\psi_1(u(t)) \leq \psi_2(u(t))$, then

(2.86) $z'(t) \leq [g(t) + h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]\, \psi_2\left(\ln(z^{\frac{1}{p}}(t))\right)$.

Hence

(2.87) $\frac{z'(t)}{\psi_2\left(\ln(z^{\frac{1}{p}}(t))\right)} \leq [g(t) + h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)].$

Integrating from a to t and making change of variable, we have

(2.88) $z^{\frac{1}{p}}(t) \leq \exp[\Psi_2^{-1}(\Psi_2(k) + G(t) + H(t))]$.

Therefore

$u(t) \leq \exp[\Psi_2^{-1}(\Psi_2(k) + G(t) + H(t))].$

The proof of case (ii) is similar.

Theorem 2.12. Let $u \in C(J, \mathbb{R}_1)$ and $g, h \in C(\mathbb{R}_+, \mathbb{R}_+)$. Suppose that $\alpha, \beta \in C^1(J, J)$ are non-decreasing with $\alpha(t) \leq t \leq \beta(t)$ on J, $u(\beta(t)) = u(\alpha(t)) = u(t), \alpha(a) = \beta(a)$. For $i = 1,2$, let $\psi_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a non-decreasing functions with $\psi_i(u) > 0$ for $u > 0$ and $k > 0$ is a constant. If the inequality

(2.89) $u(t) \leq k + \int_a^t g(s)\psi_1(\ln(u(s)))ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2(\ln(u(s)))ds,$

then for $a \leq t \leq t_1$,

(i) In case $\psi_1(u) \leq \psi_2(u)$,

(2.90) $u(t) \leq \exp[\Psi_2^{-1}[\Psi_2(k) + G(t) + H(t)]]$

(ii) In case $\psi_1(u) \geq \psi_2(u)$,

(2.91) $u(t) \leq \exp[\Psi_1^{-1}[\Psi_1(k) + G(t) + H(t)]]$

where $G(t)$ and $H(t)$ are defined by (2.13b) and (2.13c) and for $i = 1,2$, Ψ_i^{-1}, are the inverse functions of

(2.92) $\Psi_i(\mu) = \int_{\mu_0}^{\mu} \frac{ds}{\psi_i(s)}, \quad \mu > 0, \mu_0 > 0,$

and $t_1 \in J$ is chosen so that

$\Psi_i(k) + G(t) + H(t) \in Dom(\Psi_i^{-1}),$

respectively, for all t lying in the interval $[a, t_1]$.

Proof: Define a function $z(t)$ by the right- hand side of (2.62)

(2.93) $z(t) = k + \int_a^t g(s)\psi_1(\ln(u(s)))ds + \int_{\alpha(t)}^{\beta(t)} h(s)\psi_2(\ln(u(s)))ds$.

Then

(2.94) $u(t) \leq z(t)$ and $z(a) = k$.

Differentiate (2.93) we get

$$z'(t) \leq g(t)\psi_1(\ln(u(t))) + h(\beta(t))\psi_2(\ln(u(\beta(t))))\beta'(t)$$

$$-h(\alpha(t))\psi_2(\ln(u(\alpha(t))))\alpha'(t)$$

(2.95) $\leq g(t)\psi_1(z(t))) + [h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]\psi_2(\ln(z(t)))$.

Now if $\psi_1(u(t)) \leq \psi_2(u(t))$, then

(2.96) $z'(t) \leq [g(t) + h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]\,\psi_2(\ln(z(t)))$.

Hence

(2.97) $\dfrac{z'(t)}{z(t)\psi_2(\ln(z(t)))} \leq [g(t) + h(\beta(t))\beta'(t) - h(\alpha(t))\alpha'(t)]$.

Integrating from a to t and making change of variable, we have

(2.98) $z(t) \leq \exp[\Psi_2^{-1}(\Psi_2(k) + G(t) + H(t))]$.

Therefore by (2.94) we have the result. Since the proof of case (ii) is
` similar we omit the details.

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Theoretical Application of Green's Function on an Infinite Two Dimensional Hydraulic Network Pipe System

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Abstract

The equivalent hydraulic conductance between any two arbitrary junction nodes in an infinite two dimensional network system of identical pipes is calculated. The classic Lattice Green's Function method (LGF) of the Tight –Binding Hamiltonian (TBH) is generalized to infinite simple perfect square hydraulic network system of identical pipes. The equivalent conductance of hydraulic steady state flow is plotted against the junction node site. The conductance in an infinite square hydraulic network system of identical pipes is symmetric under the transformation $(l,m) \rightarrow (-l,-m)$ which is expected due to the inversion symmetry of square lattice. Results shows that the hydraulic conductance exponentially decreased in the both (l,m) and $(-l,-m)$ transformation sides (1,0) and (1,1) directions have been investigated.

Keywords: Hydraulic Conductance, Identical Pipes, Green's Function and Perfect Square Lattice.

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التطبيق النظري لدالة غرين على منظومة لا نهائية لشبكة هيدروليكية ثنائية البعد مكونة من أنابيب متماثلة

مؤيد عدالرحيم عبيد السبايلة

ملخص

تم حساب الايصالية الهيدروليكية المكافئة بين أي نقطتي وصل في منظومة لانهائية لشبكة ثنائية البعد مكونة من أنابيب متماثلة. استخدمت طريقة دالة غرين التقليدية والتي تم تعميمها على شبكة مربعة مثالية بسيطة من الأنابيب الهيدروليكية المتماثلة. تم رسم العلاقة بين الايصالية المكافئة لتدفق هيدروليكي مستقر وموقع الوصلة على الشبكة المربعة وتبين تناظرها المتوقع عند الحساب في كلا الاتجاهين. بينت كذلك النتائج أن الايصالية الهيدروليكية تتناقص آسيا على امتداد الاتجاهين (0,1) و(1,1).

الكلمات الدالة: التدفق الهيدروليكي، الأنابيب المتماثلة، دالة غرين والشبكة المربعة المثالية.

Introduction:

The steady-state analysis of flow and pressure in pipe networks is required when designing water, gas, steam or compressed air distribution systems, vacuum technology systems, chemical process plants or air conditioning systems. Pipe network analysis is therefore of great importance in engineering (Grevenstein et al., 1994; Lopes, 2004). The mathematical model consisting of volumetric flow of all possible routs of two dimensional pipe network for district hydraulic heating system was developed by Jamsek et al. (Jamsek et al., 2010). Charpin et al. obtained the model of heat removing from the construction concrete using a system of hydraulic pipe network (Charpin et al., 2004; Myers et al., 2009), where it is necessary in these network systems to find an appropriate theoretical formalism in which we can directly calculate the hydraulic conductance of flow between any two arbitrary junction nodes.

The Lattice Green Function (LGF) plays a key role in many theoretical computations, such as the phase transition in classical two-dimensional lattice coulomb gases (Lee and Teitel, 1992), the interaction between the electrons which is mediated by the phonons (Rickayzen, 1980), the effect of impurities on the transport properties of metals (Economou, 1983), the transport in inhomogeneous conductors (Kirkpatrick 1973), the resistance, capacitance and inductance calculations (Cserti, 2000; Cserti et al., 2002; Asad et al., 2006; Asad et al., 2010; Al-Sabayleh and Hijjawi, 2009). Asad et al. and Cserti et al. solved the problem in which they used LGF to calculate the resistance between any two arbitrary points in a perfect and perturbed infinite square lattice (Cserti, 2000; Cserti et al., 2002; Asad et al., 2006).

The outlines of this paper is oriented in the analytical and numerical investigations of the equivalent conductance between arbitrary junction nodes in an infinite two dimensional perfect hydraulic network system of identical pipes using Lattice Green Function. The LGF presented in this theoretical work is related to the LGF of the Tight –Binding Hamiltonian (TBH) (Economou, 1983).

Basic Definitions and Preliminaries:

Throughput Q is the basic quantity which specifies gas or steam flow. It is easily related to the particle flow rate dN/dt since, using $PV = NKT$ and $Q = P\,dV/dt$ with the temperature T fixed in this context and the pressure P constant at a particular pipe cross-section (Chambers et al., 1989; Umrath, 2007).

$$\frac{dN}{dt} = \left[\frac{P(dV/dt)}{KT}\right] = \frac{Q}{KT}$$

where K is the Boltzmann's constant = $1.38 \times 10^{-23} JK^{-1}$. It is usual to refer to the volumetric flow rates dV/dt as the gas or steam speed S (Umrath, 2007; Chambers et al., 1989).

Conductance σ measures the ease of gas or steam to flow between two regions with pressure difference $\Delta P = P_1 - P_2$, say the inlet and outlet of a pipe, between which there is a throughput Q (Umrath, 2007; Chambers et al., 1989).

$$\sigma = Q/(P_1 - P_2)$$

It clearly a sensible definition – if the same pressure drop exists across a number of pipes of different sizes then that which allows the highest throughput has the greatest conductance. Pipes of hydraulic network systems may be connected in series, parallel and mixed combination. The throughput and equivalent conductance of a steady state flow for series and parallel n pipes combination are given in table (1).

Table (1) Throughput and equivalent hydraulic conductance of a steady state flow for series and parallel n pipes combination

Series Combination		Parallel Combination	
Throughput	**Equiv. Conductance**	**Throughput**	**Equiv. Conductance**
$Q = Q_i$	$\sigma_{Eq}^{-1} = \sum_{i=1}^{n} \sigma_i^{-1}$	$Q = \sum_{i=1}^{n} Q_i$	$\sigma_{Eq} = \sum_{i=1}^{n} \sigma_i$

The conductance of a pipe of length L and diameter D for a non- turbulent flow of air at room temperature is given by Poiseuille's formula as (Chambers et al., 1989)

$$\sigma = (136 D^4 \bar{P}/L) \qquad in\ l \cdot s^{-1}$$

where $\bar{P} = (P_1 + P_2)/2$ is the average of the inlet and outlet pressures P_1 and P_2 expressed in $mbar$ and D and L are in cm.

The Analysis of Infinite Hydraulic Square System Using LGF

An infinite two dimensional hydraulic matrix consists of a number of pipes with identical conductances σ is shown in figure (1). The hydraulic pressure in this square pipe matrix at lattice point r will be denoted by $P(r)$. Then, we may write

$$\frac{Q(r)}{\sigma} = \sum_{n} [P(r) - P(r+n)] \qquad (1)$$

where n are the vectors from point r to its nearest neighbors $(n = \pm a_i, i = 1, \cdots, d)$. The right hand side of equation

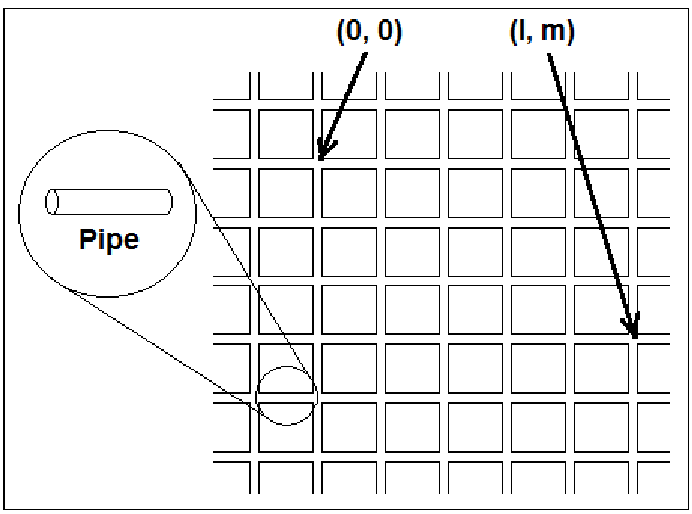

Figure (1) Infinite square hydraulic network system

(1) may be expressed by the so-called lattice Laplacian defined on the hypercubic lattice (Cserti, 2000)

$$\Delta_{(r)} f(r) = \sum_{n} [f(r+n) - f(r)] \qquad (2)$$

Thus equation (1), with the lattice Laplacian, can be rewritten as

$$\Delta_{(r)} P(r) = -[Q(r)/\sigma] \qquad (3)$$

where the throughput at lattice point r is

$$Q(r) = Q\left(\delta_{r,0} - \delta_{r,ro}\right) \qquad (4)$$

the conductance between the origin and r_o is

$$\sigma(r_o) = \frac{Q}{P(0) - P(r_o)} \qquad (5)$$

To find the conductance we solve equation (3). This is a Poisson-like equation and may be solved by using the lattice Green's function

$$P(r) = \frac{1}{\sigma}\sum_{r'} G(r - r')Q(r')$$

where the lattice Green's function is defined by (Cserti, 2000; Asad et al., 2012)

$$\Delta_{(r')} G(r - r') = -\delta_{r,r'}$$

Using equation (4) and (5), the equivalent conductance between the origin (0,0) and the junction node point (l, m) can be calculated by

$$\sigma_o(l,m) = \sigma/[G_o(0,0) - G_o(l,m)] \qquad (6)$$

the conductance between points (0,0) and (1,0) can easily be obtained as

$$\sigma_o(1,0) = \sigma/[G_o(0,0) - G_o(1,0)] \qquad (7)$$

Lattice Green Function at the site (m, n) can be expressed from integral Green's function for square lattice with nearest neighbors interaction (Cserti et al., 2002; Asad et al., 2006)

$$G(m,n,\varepsilon) = \frac{1}{\pi^2}\int_0^x\int_0^y \frac{\cos mx \cos nx}{1 - \frac{\varepsilon}{2}(\cos x + \cos y)} dxdy \qquad (8)$$

where ε is the energy parameter. By executing a partial integration with respect to x in equation (8). Therefore, the following recurrence relation (Asad et al., 2006; Alzetta et al., 1994) is obtained as

$$G'(m+1,n) - G'(m-1,n) = 2mG(m,n) \qquad (9)$$

where $G'(m,n)$ is the first derivative of $G(m,n)$ with respect to ε. Substituting (m,n) = (1,0),(1,1) and (2,0) in equation (9), respectively, the following relations are obtained

$$G'(2,0) - G'(0,0) = 2G(1,0) \qquad (10)$$

$$G'(2,1) - G'(1,0) = 2G(1,1) \qquad (11)$$

$$G'(3,0) - G'(1,0) = 4G(2,0) \qquad (12)$$

For $m = 0$ we obtain

$$\frac{4}{\varepsilon}G(0,n) - 2\delta_{0,n} - 2G(1,n) - G(0,n+1) - G(0,n-1) = 0 \qquad (13)$$

For $m \neq 0$ we have

$$G(m+1,n) - \frac{4}{\varepsilon}G(m,n) + G(m-1,n) + G(m,n+1) + G(m,n-1) = 0 \qquad (14)$$

For $n = 0$ in equation (13) we find the relation [where G(1,0) = G(0,1) = G(0,-1) due to the symmetry of the lattice, $\delta_{0,0} = 1$ and $\varepsilon = 1$]

$$G(1,0) = \frac{1}{2}\left[\frac{2}{\varepsilon}G(0,0) - 1\right] \qquad (15)$$

Thus, equation (7) becomes

$$\sigma_o(1,0) = \sigma/[G_o(0,0) - G_o(0,0) + 0.5] = 2\sigma \qquad (16)$$

To calculate the conductance between the origin and the second nearest neighbors (i.e. (1,1)) then

$$\sigma_o(1,1) = \sigma/[G_o(0,0) - G_o(1,1)] \qquad (17)$$

Substituting (m,n) = (1,0),(1,1) and (2,0) in equation (14), respectively we obtained the following relations

$$G(2,0) - \frac{4}{\varepsilon}G(1,0) + G(0,0) + 2G(1,1) = 0 \qquad (18)$$

$$G(2,1) - \frac{2}{\varepsilon}G(1,1) + G(1,0) = 0 \qquad (19)$$

$$G(3,0) - \frac{4}{\varepsilon}G(2,0) + G(1,0) + 2G(2,1) = 0 \qquad (20)$$

Using the symmetry of lattice and substituting equation (15) we obtained the following relations

$$G(1,1) = \left[\frac{2}{\varepsilon^2} - \frac{1}{2}\right]G(0,0) - \frac{1}{2}G(2,0) - \frac{1}{\varepsilon} \qquad (21)$$

$$G(2,1) = \left[\frac{4}{\varepsilon^3} - \frac{2}{\varepsilon}\right]G(0,0) - \frac{1}{\varepsilon}G(2,0) + \left[\frac{1}{2} - \frac{2}{\varepsilon^2}\right] \qquad (22)$$

$$G(3,0) = \left[\frac{3}{\varepsilon} - \frac{8}{\varepsilon^3}\right]G(0,0) + \frac{6}{\varepsilon}G(2,0) + \left[\frac{4}{\varepsilon^2} - \frac{1}{2}\right] \qquad (23)$$

Now, by taking the derivative of equation (23) with respect to ε, and using equations (10-12), the following expressions are obtained

$$G(2,0) = \left[\frac{8}{\varepsilon} - \frac{8}{\varepsilon^3}\right]G'(0,0) + G(0,0) - \frac{2}{\varepsilon} \qquad (24)$$

$$G(1,1) = \frac{1}{\varepsilon}\left[\frac{4}{\varepsilon^2} - 4\right]G'(0,0) + \left[\frac{2}{\varepsilon^2} - 1\right]G(0,0) \qquad (25)$$

$$G(2,1) = \frac{2}{\varepsilon^2}\left[\frac{4}{\varepsilon^2} - 4\right]G'(0,0) + \frac{1}{\varepsilon}\left[\frac{8}{\varepsilon^3} - 3\right]G(0,0) - \frac{1}{2} \qquad (26)$$

$$G(3,0) = \frac{12}{\varepsilon^2}\left[4 - \frac{4}{\varepsilon^2}\right]G'(0,0) + \frac{1}{\varepsilon}\left[9 - \frac{8}{\varepsilon^2}\right]G(0,0) - \left[\frac{1}{2} + \frac{8}{\varepsilon^2}\right] \qquad (27)$$

Taking the derivative of both side of equation (24) with respect to ε, and using equations (10) and (15), we obtain the following differential equation for $G(0,0)$

$$\frac{2}{\varepsilon}\left[4 - \frac{4}{\varepsilon^2}\right]G'(0,0) + \left[4 - \frac{8}{\varepsilon^2}\right]G'(0,0) - \frac{2}{\varepsilon}G(0,0) = 0 \qquad (28)$$

By using the following transformations $G(0,0) = 0.5\varepsilon\, Y(x)$ and $x = \varepsilon^2$ we obtain the following differential equation (Ashcroft and Mermin, 1976; Kittel, 1986)

$$x(1-x)\frac{d^2Y(x)}{dx^2} + (1-2x)\frac{dY(x)}{dx} - \frac{1}{4}Y(x) = 0$$

This is called the hyper geometric differential equation (Gauss's differential equation) (Asad et al., 2009). So, the solution of the differential equation is $Y(x) = (2/\pi)K(\varepsilon)$, then

$$G(0,0,\varepsilon) = \frac{\varepsilon}{2}Y(x) = \frac{\varepsilon}{\pi}K(\varepsilon) \qquad (29)$$

Using equation (29) we express $G'(0,0)$ and $G''(0,0)$ in terms of the complete elliptic integrals of the first and second kind as

$$G'(0,0,\varepsilon) = \frac{\varepsilon^2 E(\varepsilon)}{2\pi(\varepsilon^2 - 1)} \qquad (30)$$

$$G''(0,0,\varepsilon) = \frac{\varepsilon^3}{4\pi(1-\varepsilon^2)}\left[E(\varepsilon)\left(\frac{4(3-\varepsilon^2)}{\varepsilon^2}\right) - K(\varepsilon)\right] \qquad (31)$$

where $K(\varepsilon)$ and $E(\varepsilon)$ are the complete elliptic integrals of the first and second kind, respectively. The two dimensional LGF at an arbitrary junction node is obtained in closed form, which contains a sum of the complete elliptic integrals of the first and second kind. $G_o(1,1)$ can be expressed in terms of $G_o(0,0)$ and $G'(0,0)$ as

$$G_o(1,1) = \left[\frac{2-\varepsilon^2}{\varepsilon^2}\right] G_o(0,0) + \left[\frac{4-4\varepsilon^2}{\varepsilon^3}\right] G'(0,0) \qquad (32)$$

where $G_o(0,0) = (\varepsilon/\pi)K(\varepsilon)$ and

$$G_o'(0,0) = \frac{-\varepsilon^2}{4\pi}\left[\frac{E(\varepsilon)}{(1-\varepsilon)} + K(\varepsilon)\right] \qquad (33)$$

Substituting the last two expressions into equation (17), we obtain

$$\sigma_o(1,1) = (\pi/2)\sigma \qquad (34)$$

Finally, to find the conductance between the origin and any junction node (l,m) one can use the above method. Table 2 contains some conductance calculated values.

Table (2) Some calculated values of hydraulic conductance

Junction Node Site (l,m)	$(2,0)$	$(3,0)$	$(4,0)$

Conductance $\sigma_o(l, m)$	1.3761σ	1.16198σ	1.0483σ

Figure (2) and Figure (3) show the calculated values of the hydraulic conductance using recurrence formulae for the Lattice Green Function between the origin and the junction node site (l, m) of the infinite two dimensional network system along $[1,0]$ and $[1,1]$ directions. This hydraulic perfect system contains infinite number of pipes with identical conductances σ.

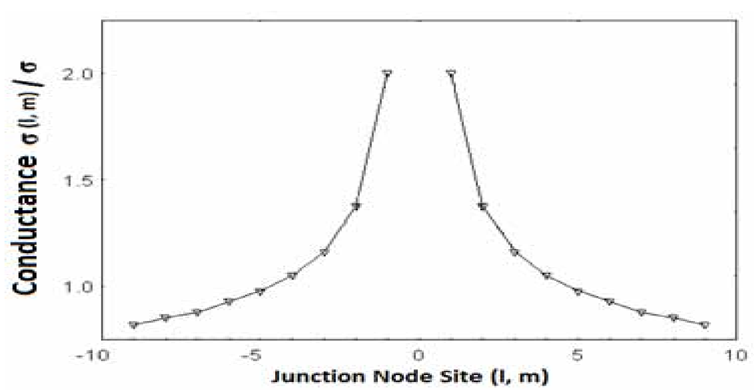

Figure (2) The calculated values of the conductance using LGF between the origin and the junction node site (l, m) of the infinite square pipe hydraulic network along $[1, 0]$ direction

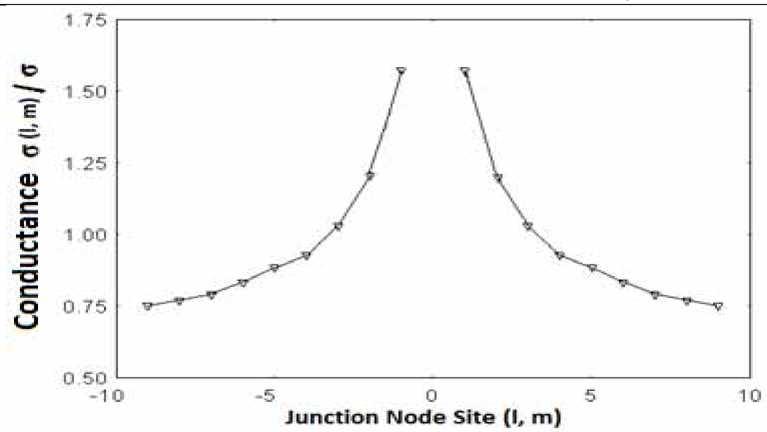

Figure (3) The calculated values of the conductance using LGF between the origin and the junction node site (l, m) of the infinite square pipe hydraulic network along $[1, 1]$ direction

From Figure (2) and Figure (3), it is obvious that curves of the calculated values of conductances along the directions (1,0) and (1,1) are symmetric. Also figures shows that the conductance exponentially decreased in the both transformation sides $(l, m) \rightarrow (-l, -m)$ of the both studied directions (1,0 and 1,1). It is shown that the conductance in (1,0) direction greater than of (1,1) direction.

Conclusions:

This paper demonstrates a theoretical approach for calculating the equivalent hydraulic conductance between two arbitrary junction nodes in an infinite square pipe network system using a Tight-Binding Hamiltonian (TBH) Lattice Green Function (LGF). Calculated results obtained using LGF recurrence formulae for an infinite perfect square network system consisting of a pipes with identical conductances are symmetric along (1,0) and (1,1) direction under the transformation $(l, m) \rightarrow (-l, -m)$ due to the inversion symmetry of the lattice. Results shows that the hydraulic conductance exponentially decreased in the both (l, m) and $(-l, -m)$ transformation sides of the both studied directions (1,0 and 1,1).

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Mu'tah Lil-Buhuth wad-Dirasat, Vol. 33 . No.1, 2018.

Approximation of Functions in Morrey Spaces by de la Vallée-Poussin Sums

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Ahmed Mohammad Kinj

Abstract

In this paper, we investigate the approximation problem of the functions by de la Vallée-Poussin sums in the Morrey space defined on the unite circle. Moreover, approximation of functions by de la Vallée-Poussin sums in Morrey-Smirnov classes defined on a simply connected domain bounded by curve satisfying Dini's smoothness condition are obtained.

Keywords: de la Vallée-Poussin; Faber polynomials; modulus of smoothness; Morrey Smirnov classes.

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تقريب الدوال بواسطة مجاميع دي فالي بوسين في فضاءات موري

محمد سليم علي
سليمان محمد محمود
أحمد محمد كنج

ملخص

قمنا في هذا العمل بدراسة تقريبات الدوال بواسطة مجاميع دي فالي بوسين في فضاء موري المعرف على دائرة الواحدة. وعلاوة على ذلك حصلنا على تقريبات الدوال بواسطة مجاميع دي فاليِ بوسين في صفوف موري سميرنوف المعرفة على منطقة بسيطة الترابط (الاتصال) محاطة بمنحنٍ يحقق شرط ديني الأملس.

الكلمات الدالة: دي فالي بوسين؛ كثيرات حدود فابير؛ معامل الملوسة؛ صفوف موري سميرنوف.

1. Introduction:

Main approximation problems in the Lebesgue spaces were studied by several authors see, for example, (Guven, 2009; Andersson, 1977). The approximation of functions of Lebesgue spaces by partial sum of Faber series was obtained by Israfilov in (Cavus & Israfilov, 1995). These results are generalized to the Muckenhoupt weighted Lebesgue's spaces in (Israfilov, 2004). Approximation properties of Faber series in weighted and non-weighted Orlicz spaces were dealt with Jafarov and Israfilov in (Jafarov, 2011; Jafarov, 2012) and (Israfilov & Akgun, 2006). In the Morrey Smirnov classes, the direct theorem of the approximation theory can be found in (Israfilov & Tozman, 2008).

Similar results in weighted Smirnov spaces and weighted Smirnov Orlicz spaces can be found in (Kokilasvili, 1968; Guven & Israfilov, 2009; Jafarov, 2016).

1- Notation and Basic definitions:

Let G be a finite simply connected domain in the complex plane $\mathbb{C}$ bounded by a rectifiable Jordan curve Γ, and let, $G^{-} := ext\,\Gamma$. Without loss of generality, we may assume that $0 \in G$. Further let $\gamma_0 := \{w \in \mathbb{C} : |w| = 1\}$, $D := int\,\gamma_0$, $D^{-} := ext\,\gamma_0$. We denote by φ the function that conformarlly and univalently maps G^{-} onto D^{-} with normalization:

$$\varphi(\infty) = \infty, \qquad \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0,$$

and let ψ be the inverse mapping of φ.

We begin with the following definitions

Definition 2.1. (Fucík, et al., 2013) For $0 \le \alpha \le 2$ and $1 \le p < \infty$. We denote by $L^{p,\alpha}(\Gamma)$ the Morrey space, as the set of locally integrable function f, with a finite norm:

$$\|f\|_{L^{p,\alpha}(\Gamma)} := \left\{ \sup_{B} \frac{1}{|B \cap \Gamma|_{\Gamma}^{1-\frac{\alpha}{2}}} \int_{B\cap\Gamma} |f(z)|^{p} |dz| \right\}^{\frac{1}{p}} < \infty, \qquad (1\text{-}1)$$

where B is an arbitrary disk centered on Γ, and $|B \cap \Gamma|_\Gamma$ is the linear Lebesgue measure of the set $B \cap \Gamma$.

In case $\Gamma = \gamma_0$ the unite circle in the complex plane, we obtain the space $L^{p,\alpha}(\gamma_0)$ for $0 \le \alpha \le 2$ and $1 \le p < \infty$.

We know that $L^{p,\alpha}(\Gamma)$ is a Banach space. If $\alpha = 2$ then the class $L^{p,2}(\Gamma)$ coincides with the class $L^p(\Gamma)$, and for $\alpha = 0$ the class $L^{p,0}(\Gamma)$ coincides with the class $L^\infty(\Gamma)$. Moreover, $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ for $0 \le \alpha_1 < \alpha_2$. Thus, $L^{p,\alpha}(\Gamma) \subset L^1(\Gamma)$, $\forall \alpha \in [0,2]$.

For $f \in L_1(\gamma_0)$ the Fourier series of f has the following formula:

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k(f) e^{ikx} = \frac{a_0}{2} + \sum_{k=0}^{\infty} a_k(f) \cos kx + b_k(f) \sin kx. \qquad (1\text{-}2)$$

Putting $A_k(x,f) = a_k(f) \cos kx + b_k(f) \sin kx$ in (1-2)

we define the $n - th$ de la Vallée-Poussin sums of series (1-2) as:

$$V_{n,m}(x,f) = \frac{1}{m+1} \sum_{k=n-m}^{n} S_k(x,f), \quad 0 \le m \le n, \qquad m,n = 1,2,3,..$$

where $S_n(x,f)$ is the $n\ th$ partial sums of Fourier series (1-2) defined by

$$S_n(x,f) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x,f) \ , n = 1,2, \dots .$$

Definition 2.2. (Devore & Lorentz, 1993) Let $0 \le \alpha \le 2$ and $1 \le p < \infty$. We define the $r\ th$ modulus of smoothness of a function $f \in L^{p,\alpha}(\gamma_0)$ for $r = 1,2,3,\dots$by the relation

$$\omega_{p,\alpha}^r(f,t) := \sup_{|h| \le t} \|\Delta_h^r(f,.)\|_{L^{p,\alpha}(\gamma_0)}, \qquad t > 0,$$

where

$$\Delta_h^r(f,x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x+kh).$$

The best approximation to $L^{p,\alpha}(\gamma_0)$ for $0 \le \alpha \le 2$ and $1 \le p < \infty$ in the class $\mathcal{T}_n$ of trigonometric polynomials of degree not greater than n is defined by

$$E_n(f)_{L^{p,\alpha}(\gamma_0)} := \inf\{\|f - T_n\|_{L^{p,\alpha}(\gamma_0)} : T_n \in \mathcal{T}_n\}.$$

If $a, b \in \mathbb{N}$ such that $a \ge b$, then we get

$$E_a(f)_{L^{p,\alpha}(\gamma_0)} \le E_b(f)_{L^{p,\alpha}(\gamma_0)}. \quad (1\text{-}3)$$

Let $f \in L^{p,\alpha}(\gamma_0)$ where $0 < \alpha < 2$, $1 < p < \infty$, $S_n(.,f)$ the $n\,th$ partial sums of Fourier series of f and T^* a trigonometric polynomial satisfies

$$E_n(f)_{L^{p,\alpha}(\gamma_0)} = \|f - T^*\|_{L^{p,\alpha}(\gamma_0)}.$$

This trigonometric polynomial exists, see for example, (Devore & Lorentz, 1993).

Using the boundedness of operator $f \to S_n(.,f)$ in the Morrey spaces $L^{p,\alpha}(\gamma_0)$ we get

$$\begin{aligned}\|f - S_n(.,f)\|_{L^{p,\alpha}(\gamma_0)} &\le \|f - T^*\|_{L^{p,\alpha}(\gamma_0)} + \|T^* - S_n(.,f)\|_{L^{p,\alpha}(\gamma_0)} \\ &= E_n(f)_{L^{p,\alpha}(\gamma_0)} + \|S_n(.,f - T^*)\|_{L^{p,\alpha}(\gamma_0)} \\ &\le E_n(f)_{L^{p,\alpha}(\gamma_0)} + C\|f - T^*\|_{L^{p,\alpha}(\gamma_0)} = (C+1)E_n(f)_{L^{p,\alpha}(\gamma_0)} \\ &= cE_n(f)_{L^{p,\alpha}(\gamma_0)},\end{aligned}$$

where $c = C + 1$, i.e. there exists a constant c such the following relation holds

$$\|f - S_n(.,f)\|_{L^{p,\alpha}(\gamma_0)} \le cE_n(f)_{L^{p,\alpha}(\gamma_0)}. \quad (1\text{-}4)$$

Definition 2.3. (Fucík, et al., 2013) We denote by $E^{p,\alpha}(G)$ the Morrey-Smirnov class, where $0 \le \alpha \le 2$ and $1 \le p < \infty$, the class of all analytic functions in G as

$$E^{p,\alpha}(G) := \{f \in E^1(G) : f \in L^{p,\alpha}(\Gamma)\}.$$

If we define $\|f\|_{E^{p,\alpha}(G)} := \|f\|_{L^{p,\alpha}(\Gamma)}$, then $E^{p,\alpha}(G)$, becomes a Banach space.

Definition 2.4 (Pommerenke, 1992) A smooth curve $\Gamma : \sigma(s)$ is called Dini-smooth, if it satisfies the condition

$$\int_0^{\delta} \frac{\Omega(\sigma'(s),s)}{s} ds < \infty, \qquad \delta > 0,$$

where $\Omega(\sigma'(s),s)$ is the modulus of continuity of function $\sigma'(s)$. By $\mathcal{D}$ we denote the set of all Dini-smooth curves.

If $\Gamma \in \mathcal{D}$, then by (Pommerenke, 1992) , it follows that

$$0 < c_1 \le |\psi'(w)| \le c_2 < \infty, \ \ 0 < c_3 \le |\varphi'(z)| \le c_4 < \infty, \qquad (1\text{-}5)$$

where c_1, c_2, c_3 and , c_4 are positive constants.

Hence, if $\Gamma \in \mathcal{D}$ and using (1-5), and (Israfilov & Tozman, 2008), we get for the function $f_0 = f \circ \psi$ the following equivalent:

$$f \in L^{p,\alpha}(\Gamma) \Leftrightarrow f_0 \in L^{p,\alpha}(\gamma_0). \qquad (1\text{-}6)$$

The function $f_0^+ : D \to \mathbb{C}$ defined by

$$f_0^+(w) = \frac{1}{2\pi i} \int_{\gamma_0} \frac{f_0(\tau)}{\tau - w} d\tau \ , w \in D, \qquad (1\text{-}7)$$

is analytic in D (Goluzin, 1969), and from (Israfilov & Tozman, 2008) we obtain
$f_0^+ \in E^{p,\alpha}(D)$.

If $\Gamma \in \mathcal{D}$ we define the $r-$ modulus of smoothness of $f \in L^{p,\alpha}(\Gamma)$ by the relation (Israfilov & Tozman, 2008)

$$\Omega^r_{\Gamma,p,\alpha}(f,t) := \omega^r_{p,\alpha}(f_0^+,t), \ \ t > 0, \ \ r = 1,2,3,\dots . \qquad (1\text{-}8)$$

The Faber polynomials $\Phi_k(z)$ of degree k are defined by the relation (Markushevich, 1968)

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{\Phi_k(z)}{w^{k+1}}, \qquad z \in G, \qquad w \in D^-. \qquad (1\text{-}9)$$

Let $f \in E^{p,\alpha}(G)$. Since $f \in E^1(G)$, by Cauchy integral, we get

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)}{s - z} ds = \frac{1}{2\pi i} \int_{\gamma_0} \frac{\psi'(w)}{\psi(w) - z} f_0(w) dw \ , z \in G.$$

From the last formula and the relation (1-9), for every $z \in G$, we get

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f)\Phi_k(z), \quad z \in G, \tag{1-10}$$

where

$$a_k(f) := \frac{1}{2\pi i}\int_{\gamma_0} \frac{f_0(w)}{w^{k+1}}\,dw, \qquad k = 0,1,2,\dots .$$

The $n-th$ partial sums, de la Vallée-Poussin sums of the series (1-10) are defined as:

$$V_{n,m}(x,f) = \frac{1}{m+1}\sum_{k=n-m}^{n} S_k(x,f), \quad 0 \le m \le n, \qquad m,n = 1,2,3,..$$

where

$$S_n(z,f) = \sum_{k=0}^{n} a_k(f)\Phi_k(z) .$$

We define the operator T as follows:

$T: E^{p,\alpha}(D) \to E^{p,\alpha}(G)$

$$T(f)(z) := \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(w)\psi'(w)}{\psi(w)-z}\,dw, z \in G. \tag{1-11}$$

In order to prove our main results, we need the following theorems.

Theorem 2.1 (Israfilov & Tozman, 2011) If $\Gamma \in \mathcal{D}$ and $L^{p,\alpha}(\Gamma)$ be a Morrey space, where $0 < \alpha \le 2$ and $1 < p < \infty$, then the linear operator $T: E^{p,\alpha}(D) \to E^{p,\alpha}(G)$ is linear, bounded one to one and onto. Moreover $T(f_0^+) = f$ for $f \in E^{p,\alpha}(G)$.

Theorem 2.2 (Israfilov & Tozman, 2008) Let $g \in E^{p,\alpha}(\mathrm{D})$ with $0 < \alpha \le 2$ and $1 < p < \infty$. Then for a given $r = 1,2,3,..$, the estimate

$$E_n(g)_{L^{p,\alpha}(\gamma_0)} \le c_5\, \omega^r_{p,\alpha}\left(g, \frac{1}{n+1}\right), n = 1,2,3,..$$

holds, with constant c_5 independent of n.

2- Main Results:

In this section, we present the main results.

Theorem 3.1 Let $L^{p,\alpha}(\gamma_0)$ be a Morrey space where $0 < \alpha \leq 2$ and $1 < p < \infty$, then there exists a positive constant c_6 such that for any $f \in L^{p,\alpha}(\gamma_0)$, $0 \leq m \leq n$, $m, n = 1,2, \dots$ the inequality

$$\|f - V_{n,m}(.,f)\|_{L^{p,\alpha}(\gamma_0)} \leq \frac{c_6}{m+1} \sum_{k=n-m}^{n} E_k(f)_{L^{p,\alpha}(\gamma_0)} \quad (2\text{-}1)$$

is true.

Corollary 3.1 Let $L^{p,\alpha}(\gamma_0)$ be a Morrey space where $0 < \alpha \leq 2$ and $1 < p < \infty$, then there exists a positive constant c_7 such that for any $f \in L^{p,\alpha}(\gamma_0)$, $0 \leq m \leq n$, $m, n = 1,2, \dots$ the inequality

$$\|f - V_{n,m}(.,f)\|_{L^{p,\alpha}(\gamma_0)} \leq \frac{c_7}{m+1} \sum_{k=n-m}^{n} \omega_{p,\alpha}^{r}\left(f, \frac{1}{k+1}\right) \quad (2\text{-}2)$$

is true.

Theorem 3.2 Let G be a simply connected domain in the complex plane bounded by curve $\Gamma \in \mathcal{D}$. If $f \in E^{p,\alpha}(G)$, where $0 < \alpha \leq 2$ and $1 < p < \infty$, then then for every $0 \leq m \leq n$, $n, m \in \mathbb{N}$ the estimate

$$\|f - V_n(.,f)\|_{L^{p,\alpha}(\Gamma)} \leq c_8 \sum_{k=n-m}^{n} \Omega_{\Gamma,p,\alpha}^{r}\left(f, \frac{1}{k+1}\right)$$

holds, where c_8 is a positive constant.

3- Proof of main results:

Proof of theorem 3.1 Let us chose the integer j such that $2^j \leq m+1 \leq 2^{j+1}$.

Then, we get

$$f(x) - V_{n,m}(x,f) = \frac{1}{m+1}[f(x) - S_{n-m}(x,f)] + \frac{1}{m+1}\left\{\sum_{k=1}^{j}\sum_{i=n-m+2^{k-1}}^{n-m+2^k-1}[f(x) - S_i(x,f)]\right\} + \frac{1}{m+1}\left\{\sum_{k=n-m+2^j}^{n}[f(x) - S_k(x,f)]\right\}.$$

And from this, we obtain

$$\|f - V_{n,m}(.,f)\|_{L^{p,\alpha}(\gamma_0)} \leq \frac{1}{m+1}\|f - S_{n-m}(.,f)\|_{L^{p,\alpha}(\gamma_0)} + \frac{1}{m+1}\left\{\sum_{k=1}^{j}\sum_{i=n-m+2^{k-1}}^{n-m+2^k-1}\|f - S_i(.,f)\|_{L^{p,\alpha}(\gamma_0)}\right\} + \frac{1}{m+1}\left\{\sum_{k=n-m+2^j}^{n}\|f - S_k(.,f)\|_{L^{p,\alpha}(\gamma_0)}\right\}. \quad (3\text{-}1)$$

From the relation (1-4), we get

$$\|f - S_{n-m}(.,f)\|_{L^{p,\alpha}(\gamma_0)} \leq cE_{n-m}(f)_{L^{p,\alpha}(\gamma_0)},$$
$$\|f - S_i(.,f)\|_{L^{p,\alpha}(\gamma_0)} \leq cE_i(f)_{L^{p,\alpha}(\gamma_0)}, \quad (3\text{-}2)$$
$$\|f - S_k(.,f)\|_{L^{p,\alpha}(\gamma_0)} \leq cE_k(f)_{L^{p,\alpha}(\gamma_0)}.$$

By using (3-1) and (3-2), we obtain

$$\begin{aligned}\|f-V_{n,m}(.,f)\|_{L^{p,\alpha}(\gamma_0)} &\le \frac{c}{m+1}E_{n-m}(f)_{L^{p,\alpha}(\gamma_0)} \\ &+\frac{c}{m+1}\left\{\sum_{k=1}^{j}\sum_{i=n-m+2^{k-1}}^{n-m+2^k-1}E_i(f)_{L^{p,\alpha}(\gamma_0)}\right\} \\ &+\frac{c}{m+1}\left\{\sum_{k=n-m+2^j}^{n}E_k(f)_{L^{p,\alpha}(\gamma_0)}\right\}.\end{aligned} \tag{3-3}$$

From the relation (1-3), we have

$$\begin{aligned}\sum_{i=n-m+2^{k-1}}^{n-m+2^k-1}E_i(f)_{L^{p,\alpha}(\gamma_0)} &\le (2^k-2^{k-1})E_{n-m+2^{k-1}}(f)_{L^{p,\alpha}(\gamma_0)} \\ &=2^{k-1}E_{n-m+2^{k-1}}(f)_{L^{p,\alpha}(\gamma_0)}.\end{aligned} \tag{3-4}$$

$$\sum_{k=n-m+2^j}^{n}E_k(f)_{L^{p,\alpha}(\gamma_0)} \le (m-2^j+1)E_{n-m+2^j}(f)_{L^{p,\alpha}(\gamma_0)}.$$

By using (3-3) and (3-4), we get

$$\begin{aligned}\|f-V_{n,m}(.,f)\|_{L^{p,\alpha}(\gamma_0)} &\le \frac{c_9}{m+1}\left\{E_{n-m}(f)_{X,\omega}+\sum_{k=1}^{j}2^{k-1}E_{n-m+2^{k-1}}(f)_{L^{p,\alpha}(\gamma_0)}\right\} \\ &+c_{10}\frac{1}{m+1}(m-2^j+1)E_{n-m+2^j}(f)_{L^{p,\alpha}(\gamma_0)}.\end{aligned} \tag{3-5}$$

On the other hand, by (1-3), we have

$$\sum_{k=1}^{j} 2^{k-1} E_{n-m+2^{k-1}}(f)_{L^{p,\alpha}(\gamma_0)}$$

$$= E_{n-m+1}(f)_{L^{p,\alpha}(\gamma_0)} + \sum_{k=2}^{j} 2^{k-1} E_{n-m+2^{k-1}}(f)_{L^{p,\alpha}(\gamma_0)}$$

$$= E_{n-m+1}(f)_{L^{p,\alpha}(\gamma_0)} + 2\sum_{k=2}^{j} 2^{k-2} E_{n-m+2^{k-1}}(f)_{L^{p,\alpha}(\gamma_0)} \qquad (3\text{-}6)$$

$$\leq E_{n-m+1}(f)_{L^{p,\alpha}(\gamma_0)} + 2\sum_{k=2}^{j} \sum_{i=n-m+2^{k-2}}^{n-m+2^{k-1}-1} E_i(f)_{L^{p,\alpha}(\gamma_0)}$$

$$\leq c_{11} \sum_{k=n-m}^{n-m+2^{j-1}} E_k(f)_{L^{p,\alpha}(\gamma_0)}.$$

Since $2^j \leq m+1 < 2^{j+1}$, we get $2^j > m - 2^j + 1$

$$(m-2^j+1)E_{n-m+2^j}(f)_{L^{p,\alpha}(\gamma_0)}$$

$$\leq 2^j E_{n-m+2^j}(f)_{L^{p,\alpha}(\gamma_0)} \sum_{k=n-m}^{n-m+2^j-1} E_k(f)_{L^{p,\alpha}(\gamma_0)}. \qquad (3\text{-}7)$$

From (3-7), and using the relation (1-3), we have

$$(m-2^j+1)E_{n-m+2^j}(f)_{L^{p,\alpha}(\gamma_0)} \leq 2^j E_{n-m+2^j}(f)_{L^{p,\alpha}(\gamma_0)}$$

$$\leq \sum_{k=n-m}^{n-m+2^j-1} E_k(f)_{L^{p,\alpha}(\gamma_0)}. \qquad (3\text{-}8)$$

From the relations (3-5), (3-6) and (3-8), we obtain

$$\|f - V_{n,m}(.,f)\|_{L^{p,\alpha}(\gamma_0)} \leq \frac{c_{12}}{m+1}\left\{E_{n-m}(f)_{L^{p,\alpha}(\gamma_0)} + \sum_{k=n-m}^{n-m+2^{j-1}} E_k(f)_{L^{p,\alpha}(\gamma_0)} + \sum_{k=n-m}^{n-m+2^j-1} E_k(f)_{L^{p,\alpha}(\gamma_0)}\right\} \leq \frac{c_{13}}{m+1}\sum_{k=n-m}^{n} E_k(f)_{L^{p,\alpha}(\gamma_0)}.$$

Thus, the inequality (2-1) is true.

Proof of Corollary 3.1 From Theorem 3.1, we have

$$\|f - V_{n,m}(.,f)\|_{L^{p,\alpha}(\gamma_0)} \leq \frac{c_6}{m+1}\sum_{k=n-m}^{n} E_k(f)_{L^{p,\alpha}(\gamma_0)}.$$

And the Theorem 2.2 give us

$$E_n(f)_{L^{p,\alpha}(\gamma_0)} \leq c\omega_{p,\alpha}^r\left(f,\frac{1}{n+1}\right), n = 1,2,3, \dots .$$

We reach to the following relation

$$\|f - V_{n,m}(.,f)\|_{L^{p,\alpha}(\gamma_0)} \leq \frac{c_{14}}{m+1}\sum_{k=n-m}^{n} \omega_{p,\alpha}^r\left(f,\frac{1}{k+1}\right), n = 1,2, \dots .$$

Proof of Theorem 3.2. Since $f \in E^{p,\alpha}(G)$ and Γ is a Dini – smooth curve, then the boundary function of f belongs to $L^{p,\alpha}(\Gamma)$ and from the relation (1-6), we get $f_0 \in L^{p,\alpha}(\gamma_0)$ and the function f_0^+ which defined by (1-7) belongs to $E^{p,\alpha}(D)$, and since $E^{p,\alpha}(D) \subset E^1(D)$, we obtain $f_0^+ \in E^1(D)$ and has the following Taylor expansion

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k(f_0^+)w^k, \quad w \in D. \qquad (3\text{-}9)$$

Let $\{c_k\}$ be the Fourier coefficients of the boundary function of f_0^+, then from (Duren, 1970) we get that $c_k = a_k(f_0^+)$ for $k \geq 0$ and $c_k = 0$ for $k < 0$ then by substitution in (3-9), we obtain

$$f_0^+(w) = \sum_{k=0}^{\infty} c_k w^k, \quad w \in D.$$

Note that for the function $f \in E^{p,\alpha}(G)$ the following Faber series holds:

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f)\Phi_k(z), \quad z \in G,$$

where $a_k(f), k = 0,1,2,\ldots$ are the Taylor coefficients of the function f_0^+ and by Theorem 2.1 we obtain

$$T\left(\sum_{k=0}^{n} a_k(f_0^+)w^k\right) = \sum_{k=0}^{n} a_k(f)\Phi_k(z),$$

and

$$T\left(V_{n,m}(\mathrm{w}, f_0^+)\right) = V_{n,m}(z,f), 0 \leq m \leq n, \qquad n,m = 0,1,2,\ldots$$

Hence using the boundedness of operator T defined by (1-11) and the relation (2-1) we reach

$$\begin{aligned}\|f - V_{n,m}(\cdot,f)\|_{L^{p,\alpha}(\Gamma)} &= \left\|T(f_0^+) - T\left(V_{n,m}(\cdot,f_0^+)\right)\right\|_{L^{p,\alpha}(\Gamma)} \\ &\leq c_{15}\left\|f_0^+ - V_{n,m}(\cdot,f_0^+)\right\|_{L^{p,\alpha}(\gamma_0)} \\ &\leq \frac{c_{16}}{m+1}\sum_{k=n-m}^{n} E_k(f_0^+)_{L^{p,\alpha}(\gamma_0)}.\end{aligned}$$

Using the Theorem 2.2, we get

$$\|f - V_{n,m}(.,f)\|_{L^{p,\alpha}(\Gamma)} \leq \frac{c_{17}}{m+1} \sum_{k=n-m}^{n} \omega_{p,\alpha}^{r}\left(f_0^+, \frac{1}{k+1}\right).$$

And by the relation (1-8), we reach

$$\|f - V_{n,m}(.,f)\|_{L^{p,\alpha}(\Gamma)} \leq \frac{c_{17}}{m+1} \sum_{k=n-m}^{n} \Omega_{\Gamma,p,\alpha}^{r}\left(f, \frac{1}{k+1}\right).$$

Consequently, we have proved the Theorem 3.2.

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Knowledge and Attitude of University Students Towards Antibiotic Self-Medication: A Cross Sectional Study at Mu'tah University

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Abstract

Introduction: Among self-medication practice, antibiotic irrational administration represents a widespread problem that leads to microbial resistance and therapy failure. Objective: Exploring the current status of antibiotic self-medication among Mu'tah University students. Methods: A descriptive cross sectional questionnaire-based study carried out involving medical and non-medical students of Mutah University. Results: Most of students reported self-medication of antibiotics for either bacterial infections or viral infections, realizing the real purpose of antibiotic administration . Switching antibiotics was observed due to lack of effect or running out of supply or side effects. Various causes for antibiotic therapy termination were reported. Conclusion: This study showed that irrational antibiotic use is customary but inappropriate, and is consistent with other studies carried out in other Jordanian Universities. The obtained results demand from health policy makers in Jordan to establish plans for preventing antibiotic misuse and to further strengthen legislations governing the sale of prescribed drugs without medical supervision.

Keywords: Antibiotic, Attitude, Antimicrobial resistance, Self-medication, University students

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معرفة وموقف طلبة الجامعة باتجاه العلاج الذاتي بالمضادات الحيوية: دراسة مقطعية مستعرضة في جامعة مؤته

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ملخص

يعتبر العلاج الذاتي بالمضادات الحيوية مشكلة منتشرة في المجتمع والتي تؤدي الى فشل علاج الأمراض الخمجية وظهور مقاومة الجراثيم المسببة لها.

الهدف من الدراسة: استكشاف الواقع الحالي للعلاج الذاتي بالمضادات الحيوية بين طلبة جامعة مؤته.

طريقة العمل: تم اجراء دراسة استبيانية مقطعية مستعرضة شملت طلبة كلية الطب وطلبة بعض الكليات غير الطبية في جامعة مؤته. النتائج: أقر معظم الطلبة باستعمالهم الذاتي للمضادات الحيوية لعلاج الأمراض البكتيرية أو الفيروسية، بالرغم من معرفتهم بالغرض الأساس لاستعمالها. ولقد استبدل المضاد الحيوي ذاتيا عند نفوذه أو عدم جدواه أو ظهور أعراضه الجانبية، كما وضحت الأسباب المؤدية لإيقافه. الاستنتاج والتوصيات: يبدوا ان الاستهلاك الذاتي للمضادات الحيوية أمر اعتاد عليه طلبة جامعة مؤته، وهذا الأمر غير لائق ولكنه ينسجم مع نتائج دراسات مماثلة في جامعات أردنية أخرى.

أن دراستنا تستهدف صانعي السياسة الصحية لمنع سوء استعمال المضادات الحيوية وتقوية التشريعات الهادفة لمنع بيعها من قبل الصيدليات بدون وصفة طبية.

Introduction:

Self-medication is defined as obtaining and consuming medication without professional supervision regarding indication, dosage, and duration of treatment (Husain and Khanum, 2008). Self-medication is a widely practiced drug misuse in both developed and developing countries. In Arab world, consumption of antibiotics to treat minor or even major diseases without medical superintendence is extensively encountered behavior. Factors contributing to widespread self-medication especially among youth were elaborated through several studies to be: socio-economical and socio-demographic factors, life style changes, drug availability and accessibility, success of self-medication, and availability of healthcare as well as health professionals (Alano et al., 2009).

Antibiotics are among the most commonly used curative drugs considered as golden bullet in modern medicine. The continuous expansion of antibiotics reflects their importance and therapeutic role which is sometimes life saving. Outweighing its effectiveness, resistance development is a real threat that broadens by time. Various factors contribute to microbial resistance, categorized into microorganism-related, and iatrogenic or patient-related. Irrational use of antibiotics is a major culprit in spread of microbial resistance (Morgan et al., 2011). In contrast to other Arabic countries, Jordanians have a higher educational level but several studies have shown inappropriate prescribing and misuse of antibiotics in Jordan (Al-Azzam et al, 2007, Sawair et al, 2009). The influence of patient beliefs, previous experience with antibiotics, pharmacy practice, governmental regulations, and physicians' improper diagnosis and treatment should be conducive to emergence of antibiotics resistance (Albsoul-Younes et al, 2010, Al-Momany et al, 2009). A variety of approaches were applied to positively manage the antibiotics resistance, namely, tailoring patient educational programs (Arnold and Straus, 2005) and improving patient awareness.

There are no available data on the current status of knowledge, attitude and behaviour regarding antibiotics use and misuse among Mu'tah or Al-Karak population in general or Mutah University students in particular, in the southern region of Jordan. The present study is aimed to evaluate this problem in a random sample of students from three different faculties at Mu'tah University.

Methods:

Study area and period

The current study was conducted in Mu'tah University located in Al-Karak governorate, south of Jordan situated at 120 km to the south of

Amman, the capital of Hashemite Kingdom of Jordan. Data was collected during the period from September to December, 2013.

The Study tool: Questionnaire and its validation

A team including faculty staff members, pharmacists and physicians developed the questionnaire used during the present study. The questionnaire was composed of 39 questions; 11 questions concerned the demographic data and the other questions dealt with the pattern of self-medication with antibiotics.

A pilot questionnaire was conducted with fifteen individuals and they confirmed its clarity and that the completion time should not exceed 10 minutes.

Students were asked to answer in writing a prepared questionnaire about self-medication with antibiotics for their presumed infections. In addition to demographic data, the questionnaire concentrated on symptoms of presumed infection for which self-medication with antibiotics was done, and the reason for this self-treatment, and the source from which antibiotics were obtained without prescription. In addition, questions were put forward about dosage of antibiotics and their modification, as well as switching of antibiotics, occurrence of side effects, and the reason for which antibiotics were stopped.

Study Design Population and Sampling:

A descriptive cross sectional study was carried out. Prior approval to the present investigative study was obtained from Mu'tah University as well as scientific and ethical committees of Faculty of Science, Faculty of Humanitarian Studies or Faculty of Medicine. The objectives of the study were explained to the participants before data collection, and their written consent was sought, and only those who filled the questionnaires were to participate. A total of 330 students participated in this study. Students who previously used antibiotics numbered 300, of whom 289 (68.5 %) students were self-medicated. Students belonged to the following Faculties: Medicine, Science, and Humanitarian studies.

Statistical Analysis:

Data collected were mathematically analyzed using SPSS software package. Statistical significance level of 0.05 was used to determine the significant association between variables applying Chi square test.

Results:

Socio-demographic analysis:

The sample students were analyzed for their age, gender, faculty, and study year. Their ages ranged from 18 – 24 years; 172 were males (52.1 %) and 158 (47.9 %) were females. Students included numbered 104 (31.5 %) from Faculty of Medicine while those from Faculty of Science or Humanitarian Studies numbered 100 (30.3 %) and 126 (38.2 %), respectively. Students were at different academic years of study, namely 67 (20.3 %), 82 (24.8 %), 115 (34.8 %), 53 (16.1 %) were from first, second, third, fourth academic years, respectively, and few students from faculty of medicine were in addition at their fifth and sixth year 2 (0.6 %) and 11 (3.3 %), respectively. The financial status of majority of students was medium (133, 40.3 %), good (134, 40.6 %), or very good (40, 12.1 %).

Antibiotic self-medication practice assessment:

Most students indicated that antibiotics are used for bacterial infections (129 students, 39.1 %) or viral infections (135 students, 40.9 %) while 28 students representing 8.5 % didn’t realize the real purpose of antibiotic administration.

The number of students who could write down the names of 3 antibiotics was 71 (68.3 %) from Faculty of Medicine (mainly from 3rd, 4th, or 6th academic years), 5 (2.9 %) from Faculty of Humanitarian Studies, and none (0%) from Faculty of Science.

The main symptoms for which antibiotic self-medication was used were runny nose, nasal congestion, fever, cough, sore throat and rarely diarrhea;

runny nose (99 students, 34.2 %) and sore throat (69 students, 23.8 %) had higher frequency scores than cough (34 students, 11.7 %), nasal congestion (27 students, 9.3 %), fever (19 students, 6.6 %), diarrhea (4 students, 1.4 %).

Self-medication was attributed to convenience (151 students, 52.2%) or lack of trust in doctors (78 students, 27 %), and to cost saving (20 students, 6.9 %).

Selection of antibiotics for self-medication was guided, in descending frequency, by opinion of family members or friends (96 students 33.2 %), recommendation by community pharmacist (69 students, 23.9 %), previous doctor prescription

(59 students, 20.9 %), or student own previous experience (52 students, 18 %) as illustrated in Figure 1 A. The factors controlling the choice of the antibiotic by students were either indication for use (92 students, 31.8 %), type of antibiotic (82 students, 28.4 %), adverse effects (56 students, 19.4 %) or brand of antibiotic

(52 students, 18 %) (Fig. 1 B). The source from which antibiotics were obtained includes community pharmacy, friends or family members, or left over from previous experience (Figure 1C); the majority of students (214 students, 74 %) obtained the antibiotic from community pharmacy.

Students checked for dosage (Table 1) either from the included package insert (98 students, 33.9 %) or by previous counseling of doctor (56 students, 19.4 %), or pharmacist (43 students, 14.9 %) or family member/friend (29 students, 10 %), or based on their previous experience (43 students, 14.9 %). Table 1 also shows that medical students consulted pharmacists, friends or family members, for antibiotic dosage significantly less than students from Faculties of Science or Humanitarian Studies. Surprisingly, self-guessing for dosage was equivalent among students from the three faculties.

Changing dose of antibiotic was done by 198 (68.5%) students and this was attributed to changing clinical condition (improvement or worsening), side effects, or insufficient supply (Figure 2 A).

Switching antibiotics was also noted by 133 (46%) students; this was due mainly to lack of effect or running out supply of antibiotic or side effects (Figure 2 B).

Switching was significantly greater ($p<0.05$) among students of Faculty of Humanitarian Studies when compared to those of Faculty of Science or Faculty of Medicine (Fig. 2 C).

The adverse effects were experienced by 112 students (38.7 %) and were mainly gastrointestinal: nausea in 75 students (25.9 %), vomiting in 49 students (19.9 %), and diarrhea in 30 students (10.4 %); skin rashes in 39 students (13.5 %), while antibiotic resistance was mentioned by 81 students (28 %). (Table 2).

Moreover, there was a significantly higher frequency of GIT side effects (nausea, vomiting and diarrhea) in male students (57 students, 19.7 %) compared to female students (48 students, 16.6 %) ($p < 0.05$). Gender differences may explain this difference in frequency of adverse effects between males and females, and can imply more caution in dose selection of antibiotics in females. Clinical resistance was high (28 %) but not significantly different in both sexes.

Stopping antibiotic use (Table 3) was done mainly when symptoms disappear (162 students, 56.1 %) or fever disappeared (48 students, 16.6 %), or supply of antibiotic ran out (35 students, 12.1 %). Table 3 also shows that 36 medical students (40.4 %) out of 89 stopped antibiotic as they got better compared to 58 students (63.7 %) out of 91 from Faculty of Science ($p< 0.05$) and 68 students (62.9 %) out of 108 from Faculty of Humanitarian Studies ($p< 0.05$).

Discussion:

These results reveal that antibiotic self-medication for presumed infection is a common phenomenon among university students, affecting about 96% of treated students. Self-medication with antibiotics was also reported in 74.6% in Greece, 59.4% in China, 19.2% in Malta and 17% in Sweden; this difference can be explained by the higher health awareness and/or the availability and the ease of antibiotic procure (Mitsi et al., 2005, Borg and Scicluna, 2002).

Our survey showed that antibiotics for self-medication were obtained mainly from community pharmacy, and were used for presumed upper respiratory infections mainly, which is also reported in a European community in 2006 (Grigoryan et al., 2006). It is useful to note here that the majority of these infections were viral in origin, and do not demand treatment with antibiotics. In a partial agreement with another Jordanian study, self-medication was mainly attributed for convenience, and sometimes lack of trust in prescribing doctors and the selection of antibiotic was guided by recommendation of pharmacist, family members or close friends, or previous personal experience including previous prescription by doctor (Suaifan et al., 2012). It is encouraged that investigators would do in foreseen future another research work related to trust of community population in doctors.

Moreover, antibiotics were switched or their dosage changed by patient according to the change in their clinical condition, occurrence of side effects, or running out of supply, which is in accordance with the results reported by Sarahroodi and Arzi (2009). The administration of multiple antibiotics and partially altered antibiotic regimen is almost equivalent to switch the antibiotic over a very short time (Sarahroodi and Arzi, 2009). The observed poor compliance with self-medication has been provided in other studies as in Greek community (Skliros et al., 2010,). Adverse reactions of antibiotic administration are significant problems. It is alarming to see that medical students, in spite of their knowledge about antibiotics and their abuse, are also the victims of this harmful self-medication phenomenon. Notable in this study is the high frequency of side effects (GI in 36.3 % of patients, skin rashes in about 13.5 %), while the frequency of clinical resistance (as explained by lack of any beneficial clinical effect of the antibiotic used) was 28 %. In comparison to a study carried out in Southern China, the occurrence of side effects is higher than that of the Chinese study (17%). This difference can be explained by incorrect dosage, cultural differences including genetic or ethnic background, availability and accessibility of antibiotics, simultaneous consumption of multiple antibiotics (Pan et al., 2012). If the situation continues as it is, resistance rates may become in due time hazardous to students and population in community of south Jordan.

Our survey clearly documented the misuse of antibiotics by undergraduate students at Mutah University. It indicates lack of proper knowledge on antibiotic use or abuse, and the main danger of misuse in encouraging further development of resistant organisms and their risk of spread in the local community. This can have serious consequences in management of acute or chronic infections.

Conclusion and Recommendations:

From the current study, the authors recommend that more efforts should be done to commence health education about the proper use and prescribing of antimicrobials, and the dangers of their misuse especially in enhancing the spread of resistant organisms in the community and its consequences in threatening life of patients with serious infections. This education must be equally conducted to university students, community doctors, and the population in the community in general.

In addition, urgency is needed to implement more restrictive regulations and laws that would penalize the sale of antibiotics without prescription.

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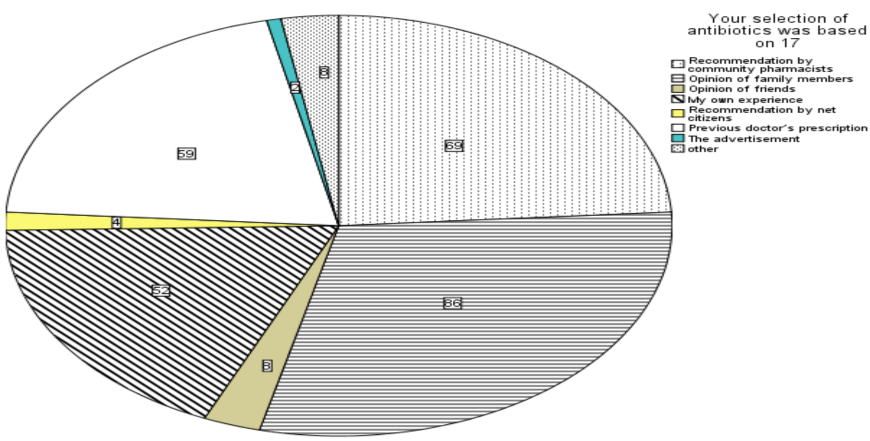

Fig 1 A : Antibiotic selection in self-medication

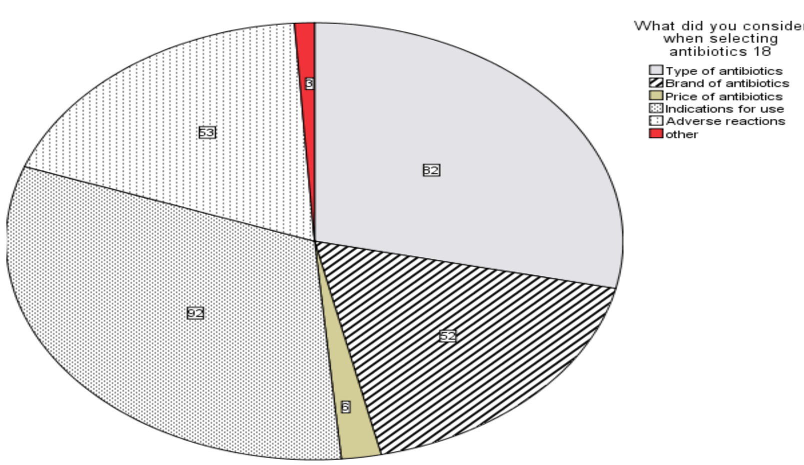

Fig 1 B : Factors considered in antibiotic selection for self-medication

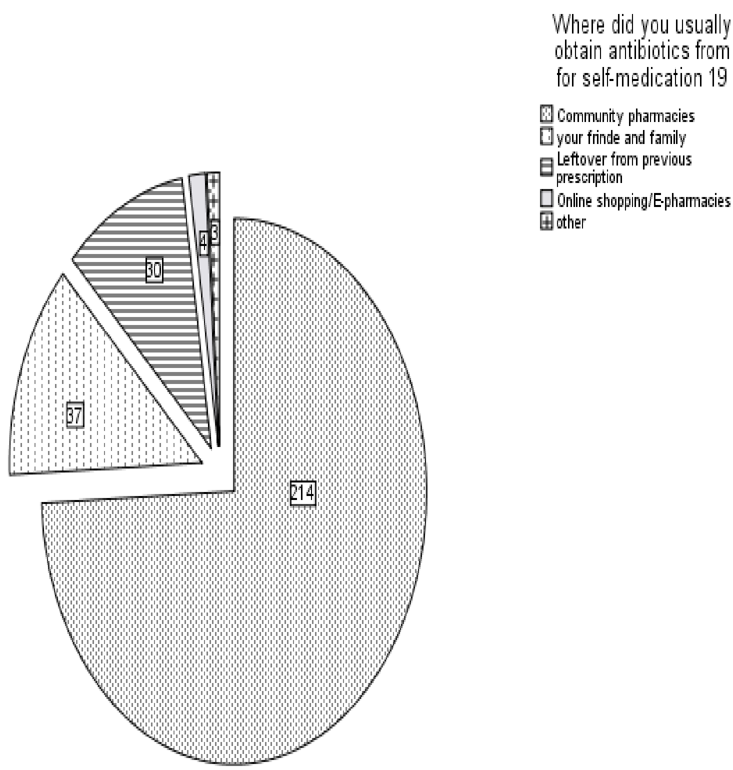

Fig 1 C : Source for obtaining antibiotics for self-medication

Table 1: Source of information for antibiotic dosage in student self-medication

Source	Students of Faculty of Science (N = 91)	Students of Faculty of Medicine (N =90)	Students of Faculty of Humanitarian Studies (N = 108)	Total N= 289
The package insert	34 (37.4%)	33 (36.7%)	31 (28.7%)	98
Doctor counseling	13 (14.3%)	21 (23.3%)*	22 (20.4%)	56
Pharmacist counseling	16 (17.6%)	8 (8.9%)**	19 (17.6%)	43
Family/friends counseling	11 (12.1 %)	7 (7.8%)*	11 (10.2%)	29
The internet	0 (0%)	2 (2.2%)	0 (0%)	2
The previous experience	13 (14.3%)	14 (15.5%)	16 (14.8%)	43
Self guessing	5 (5.5%)	5 (5.6%)	8 (7.4%)	18

* $p < 0.05$ compared to students of Faculty of Science

** $p < 0.05$ compared to students of Faculties of Science and Humanitarian Studies

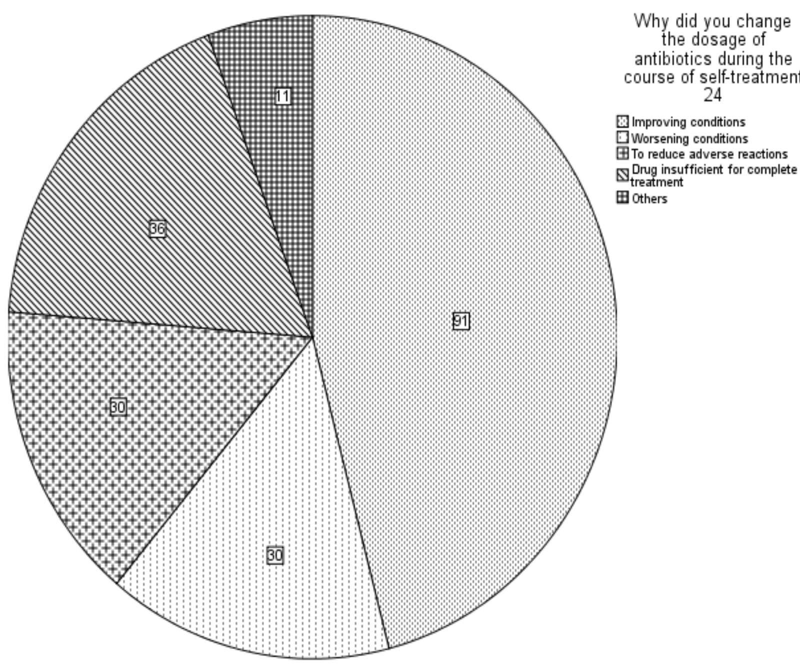

Fig 2 A : Changing dose of antibiotic during self-medication

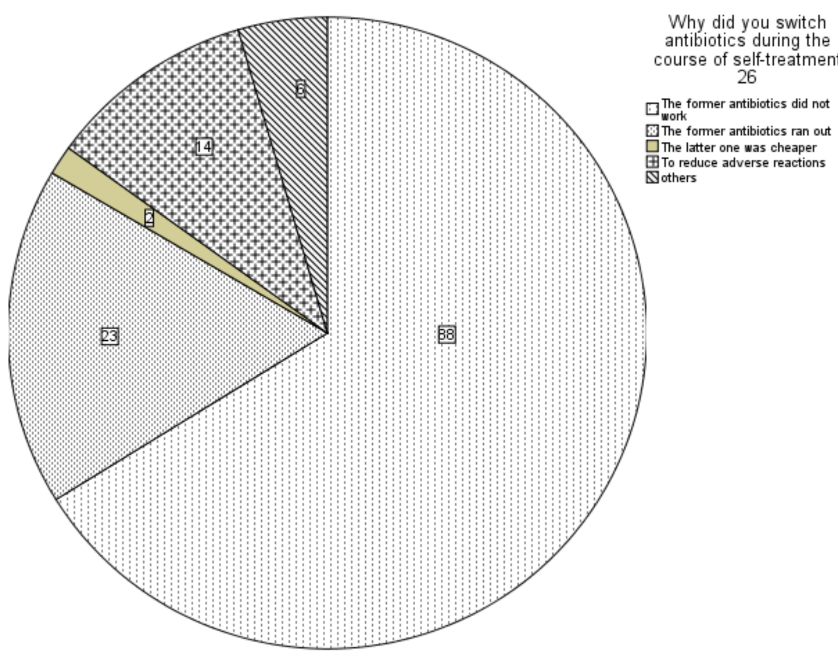

Fig 2 B : Switching antibiotic during self medication

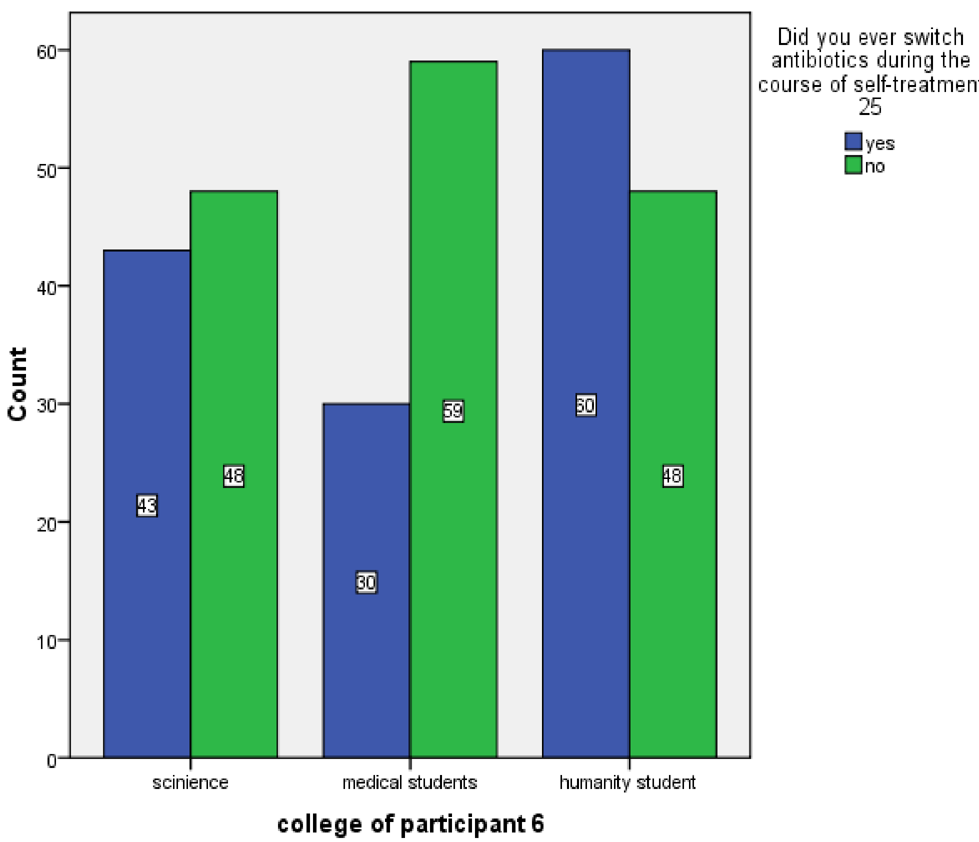

Fig. 2 C. Switching antibiotics in self-medication according to faculties.

Switching was highest among students in Faculty of Humanitarian Studies ($p < 0.05$) compared to those of Faculties of Medicine or Science.

Table (2) Adverse effects of antibiotics during self-medication in Mu'tah University students

112 Students affected out of 289 students	**Adverse effect**					
	Nausea	**Vomiting**	**Diarrhea**	**Skin rash**	**Drug resistance**	**Others**
Number of students (%)	75 25.9%	49 16.9%	30 10.4%	39 13.5%	81 28%	54 18.7%

Table (3) Stopping antibiotics in self-medication by University students

After few days regardless of outcome N (%)	After symptoms disappear N (%)	After few days of recovery N (%)	After antibiotic ran out N (%)	After complete course of treatment N (%)	After consult of doctor / pharmacist N (%)	
Faculty of Medicine (89 students)	9 (10.1%)	36 ^ (40.4%)	10* (11.2%)	22* (24,7%)	11* (12.3%)	1 (1.1%)
Faculty of Science (91students)	15 (16.5%)	58^ (63.7%)	5 (5.5%)	6 (6.6%)	6 (6.6%)	1 (1.1%)
Faculty of Humanitarian Studies (108 students)	24* (22.2%)	68*^ (63%)	3 (2.8%)	7 (6.5%)	5 (4.6%)	1 (0.9%)
Total number (288 students)	48 (16.6%)	162^ (56.3%)	18 (6.3%)	35 (12.2%)	22 (7.6%)	3 (1%)

*P<0.05 compared to students in other faculties

^ : The most frequent answer for students from the 3 faculties

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