Hamiltonian-Jacobi Treatment of Damped Harmonic Oscillator Based on Employing the Method of Dual Coordinates

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Abstract

In this paper, we provide the canonical approach for studying the damped harmonic oscillator based on the doubling of degrees of freedom approach. Explicit expressions for Lagrangians of the elementary modes of the problem and characterising both forward and backward time propagations are given. A Hamiltonian analysis showing the equivalence with the Lagrangian approach is also done. To this end, the techniques of separation of variables were applied.

Keywords: Doubling of degrees of freedom; Dissipated harmonic oscillator; Hamilton-Jacobi; Time-independent Lagrangians.

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معالجة الهزاز التوافقي المتخامد باستخدام طريقة هاميلتون- جاكوبي بعد مضاعفة درجات الحرية

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ملخص

في هذا العمل قمنا بدراسة الهزاز التوافقي المتخامد باستخدام طريقة هاميلتون- جاكوبي بعد مضاعفة درجات الحرية. لقد تم تقديم تعابير صريحة لدالة لاغرانج لوصف هذا النظام. تتضمن هذه الأنظمة صورة أساسية وصورة زمنية معكوسة. هذا يعني دائماً أن يتم وصف الأنظمة في نظام ثنائي الأبعاد. نظام فرعي واحد يبدد الطاقة والآخر يمتص الطاقة المنقولة من النظام الأول. تم تطبيق تقنيات الفصل بين المتغيرات وطريقة التحولات الفيصلية من أجل حل معادلة هاملتون جاكوبي لهذه الأنظمة.

Introduction

The Hamilton-Jacobi theory is the principal of classical mechanics. This theory principally helpful in identifying conserved quantities for mechanical systems, which may be likely even when the mechanical problem itself cannot be solved totally.

The Hamilton-Jacobi Equation is an important example of how new information about mechanics can come out of the action. It is an important step to formulate the equation of motion for simple harmonic oscillator of nth-dimensional (Goldstein, 2000).

The inverse problem of variational calculus is to construct the Lagrangian from the equations of motion. Different Lagrangian representations are obtained from the direct and indirect approaches (Santilli, 1984). In the direct representation as many variables are introduced as there are in the equations of motion. The equation of motion corresponding to a coordinate q is related with the variational derivative of the action with respect to the same coordinate. Whereas, in the indirect representation, the equation of motion is supplemented by its time - reversed image. The equation of motion with respect to the original variable then corresponds to the variational derivative of the action with respect to the image coordinate and viceversa (Bateman, 1931; Feshbach and Tikochinsky, 1977).

However, Serhan et al. obtained the action function for a suitable Hamiltonian that describes the damped harmonic oscillator, and then the system is quantized using the WKB approximation and the canonical quantization (Serhan et al., 2018). In addition, Wang and Ru Wang formulated the least action principle for classical mechanical dissipative systems. They considered a whole conservative system composed of a damped moving body and its environment receiving the dissipated energy (Qiuping Wang & Ru Wang, 2018).

Bateman showed that a procedure of doubling of degrees of freedom is required in order to use the usual canonical quantization methods (Bateman, 1931). Applying this idea to damped harmonic oscillator one obtains a pair of damped oscillations (Blasone & Jizba, 2004).This system includes a primary one and its time reversed image.

The aim of this paper is to extend the Hamilton-Jacobi formulation for doubling of degrees of freedom of the Time-Independent Mechanical systems. Furthermore, we will apply the technique of separation of variables and canonical transformations to solve the Hamilton-Jacobi partial differential equations for these systems. This leads to another approach for solving mechanical problems for doubling of degrees of freedom systems on equal footing as for regular systems.

The paper is organized as follows. In section 2, a review of the Hamilton-Jacobi Equation is introduced. In section 3, Doubling of Degrees of Freedom is discussed.Then in section 4 we present Hamilton-Jacobi Treatment of Damped harmonic Oscillator. The paper closes with some concluding remarks in section 5.

Review of the Hamilton-Jacobi Equation

We can reach the Hamilton Jacobi equation by using the action function as follows (Goldstein, 1980):

Consider the action for N particles, with 3N configuration space coordinates x_i .

$$S(x_i,t) = \int_{t_0}^{t} L(x_i,\dot{x}_i,t)dt \qquad (1)$$

The action S is defined as the integral of the Lagrangian L between two times t_0 and t.

Taking the first variation of the action integral (1) gives

$$\delta S = \sum_i \left(\left[\frac{\partial \mathcal{L}}{\partial x_i} \delta x_i \right]_{t_0}^{t} + \int_{t_0}^{t} \left(\frac{\partial \mathcal{L}}{d x_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) \delta x_i \, dt \right) \qquad (2)$$

The expression inside the integral that multiplies the variation δx_i must vanish for each i (Euler- Lagrange equations)

$$\delta S = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \delta x_i(t) = \sum_i p_i \ \delta x_i(t) \qquad (3)$$

Where we have used the assumption, that, although the particles could have started from any point x_i at the initial time, the variation of that initial point is zero : $\delta x_i(t_0) = 0$. We have also used the fact that $\frac{\partial \mathcal{L}}{\partial \dot{x}_i}$ is the momentum p_i conjugate to the coordinate x_i .

But considering the action to be a function of the final positions x_i(t) and final time t, we also have for the first variation

$$\delta S = \sum_i \frac{\partial S}{\partial \dot{x}_i}\ \delta x_i(t) \qquad (4)$$

Comparing the two expressions (3) and (4) for the first variation of the action we have

$$p_i = \frac{\partial S}{\partial x_i} \qquad (5)$$

Differentiating the action with respect to time, we obtain

$$L = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial x_i}\dot{x}_i \qquad (6)$$

Substituting equation (5) into equation (6), we obtain

$$\frac{\partial S}{\partial t} + \left(\sum_i p_i\dot{x}_i - L\right) = 0 \qquad (7)$$

The expression in brackets is recognized as the Hamiltonian $H(x_i, P_i, t)$.

Using

$$H(q_i, p_i, t) = p_i\dot{q}_i - L(q_i, \dot{q}_i, t)$$

we can write equation (7) as

$$\frac{\partial S}{\partial t} + H\left(x_i, \frac{\partial S}{\partial x_i}, t\right) = 0 \qquad (8)$$

Which is the Hamilton-Jacobi equation in terms of the action function S .

Using the canonical transformations, we may determine the coordinates x_i by differentiate the function S (x_i, t) with respect to α_i and then put the results of these differentiations to new constants β_i. In this way we obtain

$$\beta_i = \frac{\partial S}{\partial \alpha_i} \qquad (9)$$

Doubling of Degrees of Freedom

A doubling of degrees of freedom is a doubling from one field configuration to another. In dissipative systems, the energy of the damped subsystem of the whole system must be dissipated away and transferred to another subsystem. This invariably means that the damped oscillator is described by a two-dimensional system; one subsystem of which dissipates the energy and the other of which gets amplified by the transferred energy. The simplest model for dissipation is damped oscillators with one or two degrees of freedom. This kind of model has been suggested long ago by Bateman (Bateman, 1931; Hasse, 1975) and later by Morse and Feshbach (Morse and Feshbach, 1953) and Feshbach and Tikochinsky (Feshbach and Tikochinsky, 1977).

The paradigm of the dissipative dynamics is the damped harmonic oscillator model. In a one-dimensional configuration space, its equation of motion is

$$m\ddot{x} + \gamma\dot{x} + kx = 0 \quad (10)$$

Where γ is the damping constant or friction. Physically, this equation describes a classical dissipative system losing energy at a constant rate γ as time increases. This equation cannot be derived from any Lagrangian, since there is no stationary solution. In order to find out a suitable Lagrangian, one can assume that the energy lost by the system goes into another system, namely reversed–image system, which absorbs it (Morse and Feshbach, 1953).

$$m\ddot{y} - \gamma\dot{y} + ky = 0 \quad (11)$$

That is, if the energy of the oscillator described by equation (10) is lost at a rate γ, it will be gained at the same rate (with negative friction, $-\gamma$) by the reversed–image system equation (11).

This implies a zero total energy balance and, more importantly, that stationary (external) solutions for the larger system can be found.

Hamilton-Jacobi Treatment of Damped harmonic Oscillator

Homogeneous Dissipative Harmonic Oscillator

The equation of motion of the one dimensional damped harmonic oscillator is

$$m\ddot{x}+\gamma\dot{x} + kx = 0 \qquad (12)$$

where the parametersm, γ, k are time independent. However, since the system in equation (12) is dissipative, a straight forward Lagrangian description leading to a consistent canonical quantization is not available (Banerjee and Mukhejee, 2002).

In order to develop a canonical formalism, one requires equation (16) alongside its reversed image counterpart (Banerjee and Mukhejee, 2002; Ikot et al., 2010):

$$m\ddot{y} - \gamma\dot{y} + ky = 0 \qquad (13)$$

And write the variation of the action S as

$$\delta S=\int_{t_1}^{t_2} dt\ [(\frac{d}{dt}(m\dot{x}+ \gamma \Upsilon x) + kx)\delta y + (\frac{d}{dt}(m\dot{y}\ - \Upsilon\gamma\ y) + ky)\delta x \qquad (14)$$

From equation(14), equation (12) is obtained by varying S with respect to y whereas equation (13) follows from varying S with respect to x. Then the equations of motion for x and y follow as Euler – Lagrange equations for y and x respectively.

Now, starting from equation (14) we can deduce

$$\delta S = -\delta\int_{t_1}^{t_2} dt\ [\ m\dot{x}\dot{y}\ +\ \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - kxy\ ] \qquad (15)$$

It is then possible to identify

$$L = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - kxy \qquad (16)$$

The Lagrangian (16) can be written in a suggestive form by the substitution of the hyperbolic coordinates ξ and η defined by

$$x = \frac{1}{\sqrt{2}}(\xi + \eta)` \tag{17}$$

$$y = \frac{1}{\sqrt{2}}(\xi - \eta) \tag{18}$$

So that

$$L = \frac{m}{2}(\dot{\xi}^2 - \dot{\eta}^2) + \frac{\gamma}{2}(\eta\dot{\xi} - \xi\dot{\eta}) - \frac{k}{2}(\xi^2 - \eta^2) \tag{19}$$

Then the Hamiltonian reads

$$H = \frac{1}{2m}p_\xi^2 + \frac{k}{2}\xi^2 - \frac{1}{2m}p_\eta^2 - \frac{k}{2}\eta^2 \tag{20}$$

If we use the equations of transformation

$$p_\xi = \frac{\partial S}{\partial \xi} \tag{21}$$

$$p_\eta = \frac{\partial S}{\partial \eta} \tag{22}$$

We obtain the differential form of the Hamiltonian

$$H = \frac{1}{2m}\left(\frac{\partial S}{\partial \xi}\right)^2 + \frac{k}{2}\xi^2 - \frac{1}{2m}\left(\frac{\partial S}{\partial \eta}\right)^2 - \frac{k}{2}\eta^2 \tag{23}$$

The standard Hamilton-Jacobi equation for this Hamiltonian is given by

$$H + \frac{\partial S}{\partial t} = 0 \tag{24}$$

By substituting of equation (23) into equation (24), we obtain

$$\frac{1}{2m}\left(\frac{\partial S}{\partial \xi}\right)^2 + \frac{k}{2}\xi^2 - \frac{1}{2m}\left(\frac{\partial S}{\partial \eta}\right)^2 - \frac{k}{2}\eta^2 + \frac{\partial S}{\partial t} = 0 \tag{25}$$

Now we can expand the variables in the usual way of separation used in the Hamilton-Jacobi equation by assuming that S is the sum of three terms:

$$S\ (\xi, \eta, t) = W_\xi + W_\eta - \alpha_3 t \qquad (26)$$

Substituting equation (26) into equation (25), we obtain the following differential equation for W_ξ and W_η:

$$\left[\frac{1}{2m}\left(\frac{\partial W_\xi}{\partial \xi}\right)^2 + \frac{k}{2}\xi^2\right] - \left[\frac{1}{2m}\left(\frac{\partial W_\eta}{\partial \eta}\right)^2 + \frac{k}{2}\eta^2\right] - \alpha_3 = 0 \qquad (27)$$

This equation can be correct if both of the terms in the left hand side is equal to a constant, since they are functions of different variables

$$\frac{1}{2m}\left(\frac{\partial W_\xi}{\partial \xi}\right)^2 + \frac{k}{2}\xi^2 = \alpha_1 \qquad (28)$$

$$\frac{1}{2m}\left(\frac{\partial W_\eta}{\partial \eta}\right)^2 + \frac{k}{2}\eta^2 = \alpha_2 \qquad (29)$$

where α_1, α_2 and α_3 are constants such that

$$\alpha_1 - \alpha_2 = \alpha_3 \qquad (30)$$

By integrating the equations (28) and (29), we obtain

$$W_\xi = \frac{\sqrt{2m\alpha_1}}{2}\xi\sqrt{1 - \frac{m\omega^2}{2\alpha_1}\xi^2} + \frac{\alpha_1}{|\omega|} sin^{-1}\sqrt{\frac{m\omega^2}{2\alpha_1}}\xi \qquad (31)$$

and

$$W_\eta = \frac{\sqrt{2m\alpha_2}}{2}\eta\sqrt{1 - \frac{m\omega^2}{2\alpha_2}\eta^2} + \frac{\alpha_2}{|\omega|} sin^{-1}\sqrt{\frac{m\omega^2}{2\alpha_2}}\eta \qquad (32)$$

Therefore,

$$S\ (\xi,\eta,t) = \frac{\sqrt{2m\alpha_1}}{2}\xi\sqrt{1-\ \frac{m\omega^2}{2\alpha_1}\xi^2} + \frac{\alpha_1}{|\omega|}sin^{-1}\sqrt{\frac{m\omega^2}{2\alpha_1}}\ \xi +$$

$$\frac{\sqrt{2m\alpha_2}}{2}\eta\sqrt{1-\ \frac{m\omega^2}{2\alpha_2}\eta^2} + \frac{\alpha_2}{|\omega|}sin^{-1}\sqrt{\frac{m\omega^2}{2\alpha_2}}\ \eta -(\ \alpha_1-\alpha_2)\mathrm{t} \qquad (33)$$

Using the canonical transformations, we may determine the coordinates ξ and η by differentiating the function $S(\xi,\eta,t)$ with respect to α_1 and α_2 and then put the results of these differentiations to new constants β_1 and β_2 respectively. In this way we obtain

$$\beta_1 = \frac{\partial S}{\partial\alpha_1} = \frac{1}{|\omega|}sin^{-1}\sqrt{\frac{m\omega^2}{2\alpha_1}}\xi - \mathrm{t} \qquad (34)$$

$$\beta_2 = \frac{\partial S}{\partial\alpha_2} = \frac{1}{|\omega|}sin^{-1}\sqrt{\frac{m\omega^2}{2\alpha_2}}\eta + \ \mathrm{t} \qquad (35)$$

equations (34) and (35) can be solved to give

$$\xi = \sqrt{\frac{2\alpha_1}{m\omega^2}}\ \sin\ (\omega\beta_1 + \omega t) \qquad (36)$$

$$\eta = \sqrt{\frac{2\alpha_2}{m\omega^2}}\ \sin\ (\omega\beta_2 - \omega t) \qquad (37)$$

To determine the momenta p_ξ and p_η, we may differentiate the function $S(\ \xi,\eta,t)$ with respect to ξ and η. In this way we obtain

$$p_\xi = \frac{\partial S}{\partial\xi} = \sqrt{2m\alpha_1 -\ m^2\omega^2\xi^2} \qquad (38)$$

$$p_\eta = \frac{\partial S}{\partial x_2} = \sqrt{2m\alpha_2 -\ \ m^2\omega^2\eta^2} \qquad (39)$$

Substituting equations (36) and (37) into equations (38) and (39); respectively, we obtain

$$p_\xi = \sqrt{2m\alpha_1}\ \cos\ (\omega\beta_1 + \omega t) \qquad (40)$$

$$p_{\eta}= \sqrt{2m\alpha_2}\ \cos(\omega\beta_2 - \omega t) \qquad (41)$$

Substituting the equations (36) and (37) into equations (17) and (18), we get the final results of the equations of motion in terms of time:

$$x = \frac{1}{\sqrt{2}}(\xi + \eta) = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2\alpha_1}{m\omega^2}} sin(\omega\beta_1 + \omega t) + \sqrt{\frac{2\alpha_2}{m\omega^2}} sin(\omega\beta_2 - \omega t)\right] \qquad (42)$$

$$y = \frac{1}{\sqrt{2}}(\xi - \eta) = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2\alpha_1}{m\omega^2}} sin(\omega\beta_1 + \omega t) - \sqrt{\frac{2\alpha_2}{m\omega^2}} sin(\omega\beta_2 - \omega t)\right] \qquad (43)$$

One can notice that the damping factor γ is no longer exist through all the results obtained.

Non-Homogeneous Dissipative Harmonic Oscillator

The equation of motion of the one dimensional damped harmonic oscillator is

$$\ddot{x} - 2\Upsilon\,\gamma\,\dot{x} + \omega^2 x = \mathcal{E} y \qquad (44)$$

where the parameters γ, ω, $\mathcal{E}$ are time independent. However, since the system in equation (44) is dissipative, a straight forward Lagrangian description leading to a consistent canonical quantization is not available (Banerjee and Mukhejee, 2002).

In order to develop a canonical formalism, one requires equation (44) alongside its reversed image counterpart (Banerjee and Mukhejee, 2002; Banerjee andMukherjee, 2015):

$$\ddot{y} + 2\,\gamma\Upsilon\dot{y} + \omega^2 y = \mathcal{E} x \qquad (45)$$

The first oscillator is represented by x whereas the second oscillator is represented by y.

The special case $\mathcal{E}= 0$ is the uncoupled motion of the two oscillators that corresponds to Bateman’s doublet consisting of a damped harmonic oscillator and its time reversed image (Bateman, 1931; Caldirola, 1941; Kanai, 1948).

The Lagrangian of the system can be constructed by the inverse Lagrangian method. First, we write the variation of the action as

$$\delta S= \int_{t_1}^{t_2} dt\ [(\frac{d}{dt}(\dot{x} - 2\Upsilon\,\gamma x) + \omega^2 x - \mathcal{E} y)\delta y + (\frac{d}{dt}\ (\ \dot{y}\ + 2\,\gamma\Upsilon y) + \omega^2 y - \mathcal{E} x)\delta x \quad (46)$$

From equation (46), equation(44) is obtained by varying S with respect to y whereas equation (45) follows from varying S with respect to x. Then the equations of motion for x and y follow as Euler – Lagrange equations for y and x respectively.

Now, starting from equation (46) we can deduce that

$$\delta S = -\delta\int_{t_1}^{t_2} dt\ [\ \dot{x}\dot{y} - \gamma\,\Upsilon(x\dot{y} - \dot{x}y) - \omega^2 xy\ + \frac{\mathcal{E}}{2}(x^2 + y^2)] \qquad (47)$$

It is then possible to identify

$$L = \dot{x}\dot{y} - \Upsilon\gamma\ (x\dot{y} - \dot{x}y) - \omega^2\ x\ y + \frac{\mathcal{E}}{2}(x^2 +\ y^2) \qquad (48)$$

The Lagrangian (48) can be written in a suggestive form by the substitution of the hyperbolic coordinates ξ and η defined by

$$x = \frac{1}{\sqrt{2}}(\ \xi + \eta) \qquad (49)$$

$$y = \frac{1}{\sqrt{2}}(\ \xi - \eta) \qquad (50)$$

The Lagrangianis then

$$L= \frac{1}{2}(\ \dot{\xi}^2 - \dot{\eta}^2) -\ \Upsilon(\ \eta\dot{\xi} - \xi\dot{\eta})\ - \frac{\omega^2}{2}(\ \xi^2 - \eta^2) + \frac{\mathcal{E}}{2}(\xi^2 + \eta^2) \qquad (51)$$

we find that the Hamiltonian H reeds

$$H = \frac{1}{2}p_\xi^2 - \frac{1}{2}p_\eta^2 + \frac{1}{2}(\omega^2 -\ \mathcal{E})\ \xi^2 -\ \frac{1}{2}(\omega^2 +\ \mathcal{E})\eta^2 \qquad (52)$$

If we use the equations of transformation

$$p_\xi = \frac{\partial S}{\partial \xi} \qquad (53)$$

$$p_\eta = \frac{\partial S}{\partial \eta} \qquad (54)$$

We obtain the differential form of the Hamiltonian

$$H = \frac{1}{2}\left(\frac{\partial S}{\partial \xi}\right)^2 + \frac{1}{2}(\omega^2 - \mathcal{E})\,\xi^2 - \frac{1}{2}\left(\frac{\partial S}{\partial \eta}\right)^2 - \frac{1}{2}(\omega^2 + \mathcal{E})\eta^2 \qquad (55)$$

The standard Hamilton-Jacobi equation for this Hamiltonian is given by

$$H + \frac{\partial S}{\partial t} = 0 \qquad (56)$$

By substituting of (55) in (56), we obtain

$$\frac{1}{2}\left(\frac{\partial S}{\partial \xi}\right)^2 + \frac{1}{2}(\omega^2 - \mathcal{E})\,\xi^2 - \frac{1}{2}\left(\frac{\partial S}{\partial \eta}\right)^2 - \frac{1}{2}(\omega^2 + \mathcal{E})\eta^2 + \frac{\partial S}{\partial t} = 0 \qquad (57)$$

Now we can expand the variables in the usual way of separation used in the Hamilton-Jacobi equation by assuming that S is the sum of three terms:

$$S(\xi, \eta, t) = W_\xi + W_\eta - \alpha_3 t \qquad (58)$$

By substituting of equation (58) into equation (57) we obtain the following differential equation for W_ξ and W_η

$$\left[\frac{1}{2}\left(\frac{\partial W_{\mathcal{E}}}{\partial \xi}\right)^2 + \frac{1}{2}(\omega^2 - \mathcal{E})\,\xi^2\right] - \left[\frac{1}{2}\left(\frac{\partial W_\eta}{\partial \eta}\right)^2 + \frac{1}{2}(\omega^2 + \mathcal{E})\eta^2\right] - \alpha_3 = 0 \qquad (59)$$

This equation must be correct if both of the terms in the left hand side are equal to a constant, since they are functions of different variables

$$\frac{1}{2}\left(\frac{\partial W_{\mathcal{E}}}{\partial \xi}\right)^2 + \frac{1}{2}(\omega^2 - \mathcal{E})\,\xi^2 = \alpha_1 \qquad (60)$$

$$\frac{1}{2}\left(\frac{\partial W_\eta}{\partial \eta}\right)^2 + \frac{1}{2}(\omega^2 + \mathcal{E})\eta^2 = \alpha_2 \qquad (61)$$

Where α_1 , α_2 and α_3 are the constants such that

$$\alpha_1 - \alpha_2 = \alpha_3 \qquad (62)$$

By integrating the equations (60) and (61), we obtain

$$W_\xi = \frac{\sqrt{2\alpha_1}}{2}\,\xi\sqrt{1 - \left(\frac{\omega^2 - \mathcal{E}}{2\alpha_1}\right)\ \xi^2} + \frac{\alpha_1}{\sqrt{(\omega^2 - \mathcal{E})}}\ \sin^{-1}\sqrt{\frac{\omega^2 - \mathcal{E}}{2\alpha_1}}\ \xi \qquad (63)$$

And

$$W_\eta = \frac{\sqrt{2\alpha_2}}{2}\eta\sqrt{1-\frac{(\omega^2+\varepsilon)}{2\alpha_2}\eta^2} + \frac{\alpha_2}{\sqrt{\omega^2+\varepsilon}}\ sin^{-1}\sqrt{\frac{\omega^2+\varepsilon}{2\alpha_2}}\eta \qquad (64)$$

Therefore,

$$S(\xi,\eta,t) = \frac{\sqrt{2\alpha_1}}{2}\xi\sqrt{1-\frac{(\omega^2-\varepsilon)}{2\alpha_1}\xi^2} + \frac{\alpha_1}{\sqrt{(\omega^2-\varepsilon)}}\ sin^{-1}\sqrt{\frac{(\omega^2-\varepsilon)}{2\alpha_1}}\ \xi\ + \frac{\sqrt{2\alpha_2}}{2}\eta\sqrt{1-\frac{(\omega^2+\varepsilon)}{2\alpha_2}\eta^2} + \frac{\alpha_2}{\sqrt{(\omega^2+\varepsilon)}}\ sin^{-1}\sqrt{\frac{(\omega^2+\varepsilon)}{2\alpha_2}}\eta - (\alpha_1-\alpha_2)t \qquad (65)$$

Using the canonical transformations, we may determine the coordinates ξ and η by differentiating the function S(ξ ,η ,t) with α_1 and α_2 and then put the results of these differentiations to new constants $\beta_1$and β_2, respectively. In this way we obtain

$$\beta_1 = \frac{\partial S}{\partial\alpha_1} = \frac{1}{\sqrt{(\omega^2-\varepsilon)}}\ sin^{-1}\sqrt{\frac{\omega^2-\varepsilon}{2\alpha_1}}\ \xi - t \qquad (66)$$

$$\beta_2 = \frac{\partial S}{\partial\alpha_2} = \frac{1}{\sqrt{(\omega^2+\varepsilon)}}\ sin^{-1}\sqrt{\frac{\omega^2+\varepsilon}{2\alpha_2}}\eta + t \qquad (67)$$

Equations (66) and (67) can be solved to give

$$\xi = \sqrt{\frac{2\alpha_1}{(\omega^2-\varepsilon)}}\, sin\sqrt{(\omega^2-\varepsilon)}\ (\beta_1 + t) \qquad (68)$$

$$\eta = \sqrt{\frac{2\alpha_2}{(\omega^2+\varepsilon)}}\, sin\sqrt{(\omega^2+\varepsilon)}\ (\beta_2 - t) \qquad (69)$$

To determine the momenta p_ξ and p_η we may differentiate the function S(ξ,η,t) with respect to ξ and η. In this way we obtain

$$p_\xi = \frac{\partial S}{\partial\xi} = \sqrt{2\alpha_1 - (\omega^2-\varepsilon)\,\xi^2} \qquad (70)$$

$$p_\eta = \frac{\partial S}{\partial\eta} = \sqrt{2\alpha_2 - (\omega^2+\varepsilon)\eta^2} \qquad (71)$$

Substituting equations (68) and (69) into equations (70) and (71); respectively, we obtain

$$p_\xi = \sqrt{2\alpha_1}\cos(\sqrt{(\omega^2 - \varepsilon)}(\beta_1 + t)) \quad (72)$$

$$p_\eta = \sqrt{2\alpha_2}\cos(\sqrt{(\omega^2 + \varepsilon)}(\beta_2 - t)) \quad (73)$$

Substituting the equations (68) and (69) into equations (49) and (50), we get the final results of the equations of motion in terms of time:

$$x = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2\alpha_1}{\omega^2-\varepsilon}}\,sin\,(\sqrt{(\omega^2-\varepsilon)}(\beta_1+t)) + \sqrt{\frac{2\alpha_2}{\omega^2+\varepsilon}}\,sin\,(\sqrt{(\omega^2+\varepsilon)}(\beta_2 - t))\right] \quad (74)$$

$$y = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{2\alpha_1}{\omega^2-\varepsilon}}\,sin(\sqrt{(\omega^2-\varepsilon)}(\beta_1+t) - \sqrt{\frac{2\alpha_2}{\omega^2+\varepsilon}}\sin\,(\sqrt{(\omega^2+\varepsilon)}(\beta_2 - t))\right] \quad (75)$$

One can notice that the damping factor γ is no longer exist through all the results obtained.

Figures (1) and (2) show that the system described by the variable x releases its energy (dissipation) while the system described by the variable y absorbs the energy from the system described by the x variable (Majima and Suzuki, 2011).

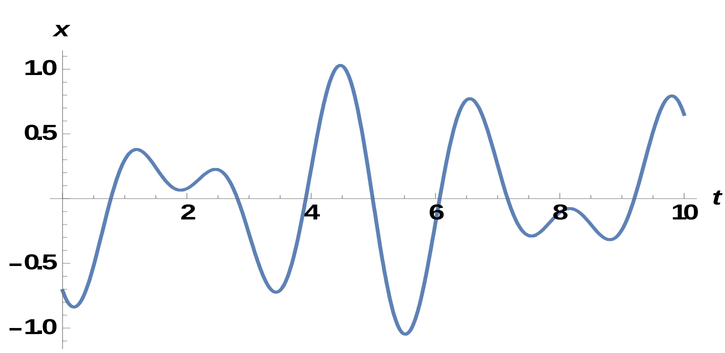

Fig.1: The coordinate x as a function of time t

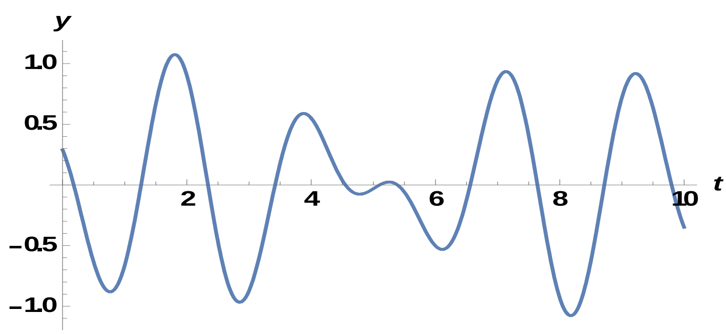

Fig.2: The coordinate y as a function of time t

Conclusion

This work has aimed to study Hamiltonian-Jacobi Method of Time-Independent Mechanical Systems Based on the Doubling of Degrees of Freedom. One can double the degrees of freedom in order to use the usual canonical transformation methods. Applying this idea to three examples of harmonic oscillator, one obtains a pair of damped oscillations a primary one and its time reversed image ($t \rightarrow -t$).

Any formulation of the harmonic oscillator is based on the direct or indirect representation. The direct representation leads to lagrangians having an explicit time dependence; hence these are not very popular. The indirect representation avoids this problem by a doubling of the degrees of freedom. It is called indirect because, taking the composite Lagrangian and varying one degree of freedom yields the equation of motion for the other degree. The usual composite Lagrangian, by construction, is two dimensional. It incorporates both forward and backward time propagations.

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