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Results of Special Types for Subclasses of Analytic Functions

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Abstract

A subclass of normalized analytic functions symmetric with respect to points is introduced. The second, the third and the fourth coefficients bounds are obtained. The Fekete-Szego inequality for a univalent normalized functions f in the subclass $SK_s(\lambda, A, B)$ is obtained and Hankal determinant inequalities are estimated. The solution $w = f(z) = \beta \exp\left(\frac{\lambda}{\lambda-1}\right)$ Is given for some values of a constant β for the differential equation $\lambda \frac{\partial f(z)}{\partial z}\ \varphi_1\left(\frac{\partial f(z)}{\partial z}\right) + (1-\lambda)\frac{\partial (zf'(z))}{\partial z}\ \varphi_0(f(z)) = 0.$

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نتائج لأنـــــواع خاصة من عوائـــل اقترانات تحليلـــية

فيصل الكساسبة

ملخص

قدمت عائلة من الاقترانات التحليلية المحددة ذات النقاط المتماثلة. وتم التوصل الى حدود المعامل الثاني والثالث وحسبت ومتباينة محدد هانكل. كما قدم الحل $SK_s(\lambda, A, B)$. متباينة فيكت− سزيقو للاقترانات الاحادية التحليلية في العائلة

في العادلة التفاضلية لقيم β $w = f(z) = \beta \exp\left(\frac{\lambda}{\lambda - 1}\right)$

$$\lambda \frac{\partial f(z)}{\partial z} \varphi_1\left(\frac{\partial f(z)}{\partial z}\right) + (1 - \lambda) \frac{\partial (z f'(z))}{\partial z} \varphi_0\big(f(z)\big) = 0$$

***Keywords**: Starlike functions with respect to points; subordination; Carathéodory function; Fekete-Szego inequality.

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Introduction

Let $\mathcal{A}$ denote the class of allnormalized analytic functions in unit disk $\mathcal{D} = \{z : |z| < 1\}$ in the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, where $a_n \in \mathbb{C}$. And let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. For two functions $f, g \in A$, we say f is subordinate to g in $\mathcal{D}$, written as $f \prec g$or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ with $w(0) = 0$ *and* $|w(z)| < 1$ such that $f(z) = g(w(z))$ for $z \in \mathcal{D}$. Moreover, if g is univalent in $\mathcal{D}$ then; $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{D}) \subset g(\mathcal{D})$.

In 1959, Sakaguchi(1959) introduced a subclass $\mathcal{S}_s^*$ of starlike functions with respect to symmetric points as follows: A function $f \in \mathcal{S}_s^*$ is starlike with respect to symmetric points if

$$Re\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0 \quad , \text{ for all } z \in \mathcal{D}.$$

In 1977, Das and Singh(1977) introduced other subclasses of convex functions with respect to symmetric points as follows: A function $f \in K_s$ is convex with respect to symmetric points if

$$Re\left(\frac{(zf'(z))'}{(f(z)-f(-z))'}\right) > 0, \quad \text{for } all \ z \in \mathcal{D}.$$

After that many authors such as Owa et al. (1988), Wu(1987) and (Sakaguchi, 1959, Goel, & Mehrok, 1982., Abdul Halim, 1999., Mishra et al., 2016) discussed results of the above subclasses. In 1982, Goel and Mehrok introduced the subclass $\mathcal{S}_s^*(A,B)$ of $\mathcal{S}_s^*$, in term of subordination.

Definition 1.1(Goel, & Mehrok, 1982) An analytic function $f \in \mathcal{A}$ is said to be belonging to the subclass $\mathcal{S}_s^*(A,B)$, if

$$\frac{zf'(z)}{f(z)-f(-z)} \prec \frac{1+Az}{1+Bz} \quad , \text{ for } -1 \le B < A \le 1 \text{ and } z \in \mathcal{D}. \tag{1.1}$$

Definition 1.2(Selvaraj & Vasanthi, 2011) An analytic function $f \in \mathcal{A}$ is said to be belonging to the subclass $K_s(A,B)$, if

$$\frac{(zf'(z))'}{(f(z)-f(-z))'} \prec \frac{1+Az}{1+Bz} \quad , \text{for} -1 \le B < A \le 1 \text{and} z \in \mathcal{D}. \tag{1.2}$$

Definition 1.3(Ravichandran , 2004). A function $f \in \mathcal{A}$ is in the class $\mathcal{S}_s^*(\xi)$, if

$$\frac{zf'(z)}{f(z)-f(-z)} \prec \xi(z) = 1 + \xi_1 z + \xi_2 z^2 + \cdots \quad , \ for\ z \in \mathcal{D}. \tag{1.3}$$

And $a\ function\ f \in \mathcal{A}\ is\ in\ the\ class\ K_s(\xi)$, if

$$\frac{(zf'(z))'}{(f(z)-f(-z))'} \prec \xi(z) = 1 + \xi_1 z + \xi_2 z^2 + \cdots \quad , \quad for\ z \in \mathcal{D}. \tag{1.4}$$

In this paper, the operator φ_n of a function f is defined to be;

$$\varphi_n(f(z)) = \frac{f^{(n)}(z)-(-1)^n f^{(n)}(-z)}{2}, \qquad \text{where} \qquad n \in \{0,1,2,3,\dots\}.$$

In particular, if $n = 0$ then $\varphi_0(f(z)) = \frac{f(z)-f(-z)}{2} = z + \sum_{n=1}^{\infty} a_{2n+1} z^{2n+1}$.

Definition1.4. An analytic function $f \in \mathcal{A}$ is said to be belonging to the subclass $SK_s(\lambda, A, B)$, if

$$\lambda \frac{zf'(z)}{\varphi_0(f(z))} + (1-\lambda)\frac{(zf'(z))'}{\varphi_1(f(z))} \prec \frac{1+Az}{1+Bz}, \tag{1.5}$$

for $\lambda \in [0,1]$, $-1 \le B < A \le 1$ and $z \in \mathcal{D}$.

Definition1.5. An analytic function $f \in \mathcal{A}$ is said to be belonging to the subclass $SK_s(\lambda, \xi)$, if

$$\lambda \frac{zf'(z)}{\varphi_0(f(z))} + (1-\lambda)\frac{(zf'(z))'}{\varphi_1(f(z))} \prec \xi(z) = 1 + \xi_1 z + \xi_2 z^2 + \cdots , \tag{1.6}$$

It is clear that if $\lambda = 1$ then $SK_s(1, A, B) = \mathcal{S}_s^*(A, B)$, and if $\lambda = 0$ then $SK_s(0, A, B) = K_s(A, B)$. So that $\mathcal{S}_s^*(A, B)$ and $K_s(A, B)$ are two subclasses of $SK_s(\lambda, A, B)$.

Furthermore, Let $\mathcal{P}$ denote the class of functions $\xi(z)$ of the form

$$\xi(z) = 1 + \sum_{n=1}^{\infty} \xi_n z^n \quad , \ for\ all\ z \in \mathcal{D} \tag{1.7}$$

Which are analytic in $\mathcal{D}$. If $\xi(z) \in \mathcal{P}$ satisfies $Re\,\xi(z) > 0 (z \in \mathcal{D})$, then we say that $\xi(z)$ is the Carathéodory function and $\xi(z) = \frac{1+Az}{1+Bz}$, for $-1 \leq B < A \leq 1$.

For $A, B \in \mathbb{C}$ with $A \neq B$, $|B| = 1, |A| \leq 1$, $Re(A\bar{B}) < 1$ the function $\xi(z)$ is analytic univalent in $\mathcal{D}$, and $Re(\xi(z)) > \frac{1-|A|^2}{2(1-A\bar{B})} \geq 0$.

If $f \in SK_s(\lambda, A, B)$ and $\xi(\omega(z)) = \frac{1+A\omega(z)}{1+B\omega(z)}$ then

$$\lambda \frac{zf'(z)}{\varphi_0(f(z))} + (1-\lambda)\frac{(zf'(z))'}{\varphi_1(f'(z))} = \frac{1+A\omega(z)}{1+B\omega(z)} = \xi(z).$$

Lemma 1.1(Goel & Mehrok, 1982) .If ξ is given by (1.7), then $|\xi_n| \leq (A-B)$, $for\ n = 1,2,3, \ldots$

Lemma 1.2. (Pommerenke, 1975). If ξ is given by (1.7), then $|\xi_n| \leq 2$, $for\ each\ n$.

Lemma 1.3.(Ma & Minda, 1994). If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic function with positive real part in $\mathcal{D}$, then

$$|c_2 - \nu c_1^2| = \begin{cases} -4\nu + 2 & if\ \nu \leq 0 \\ 2 & if\ 0 \leq \nu \leq 1 \\ 4\nu - 2 & if\ 1 \leq \nu \end{cases}.$$

When $\nu > 1$ or $\nu < 0$, the equality holds if and only if $p_1(z) = \frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p_1(z) = \frac{1+z^2}{1-z^2}$. If $\nu = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z}\ ,\quad for\ (0 \leq \gamma \leq 1)$$

or one of its rotations. If $\nu = 1$, the equality holds when the reciprocal of one of the functions such that the equality holds in the case of $\nu = 0$. Also the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$|c_2 - \nu c_1^2| + \nu|c_1|^2 \leq 2 \quad (0 < \nu \leq \frac{1}{2})$$

And

$$|c_2 - \nu c_1^2| + (1-\nu)|c_1|^2 \leq 2 \quad (\frac{1}{2} < \nu < 1).$$

The main results

We obtain the coefficients bounds for a function f belong to the subclass $SK_s(\lambda, A, B)$.

Theorem 2.1 Let $\xi(z) = 1 + \xi_1 z + \xi_2 z^2 + \cdots$ *and* $f \in SK_s(\lambda, \xi)$. Then

$$|a_2| \le \frac{\xi'(0)}{2(2-\lambda)}$$

$$|a_3| \le \frac{\xi''(0)}{4(3-2\lambda)}$$

$$|a_4| \le \frac{1}{4(4-3\lambda)}\left(\frac{\xi''(0)\xi'(0)}{2(3-2\lambda))} - \frac{(2+\lambda)\xi''(0)\xi'(0)}{8(2-\lambda)(3-2\lambda)} + \frac{\xi'''(0)}{6}\right).$$

Proof.

Since $\lambda\, zf'(z)\, \varphi_1(f'(z)) + (1-\lambda)(zf'(z)'\, \varphi_0(f(z)) = \xi(z)\, \varphi_0(f(z))\, \varphi_1(f'(z))$

Then

$$\lambda\left(z + \sum_{n=2}^{\infty} n\, a_n z^n\right)\left(1 + \sum_{n=1}^{\infty}(2n+1)a_{2n+1} z^{2n}\right) +$$

$$(1-\lambda)(1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1})(z + \sum_{n=1}^{\infty} a_{2n+1} z^{2n+1}) =$$

$$\left(1 + \sum_{n=1}^{\infty} \xi_n z^n\right)\left(z + \sum_{n=1}^{\infty} a_{2n+1} z^{2n+1}\right)(1 + \sum_{n=1}^{\infty}(2n+1)a_{2n+1} z^{2n})$$

Then

$$4a_2 - 2a_2\lambda = \xi'(0), 12\lambda a_3 + 20a_3(1-\lambda) = \xi''(0) + 8a_3$$

And $64a_4 + 12\lambda a_3 a_2 + 24 a_3 a_2 - 72\lambda a_4 = 24 a_3 \xi'(0) + \xi'''(0).$

Thus

$|a_2| \le \frac{\xi'(0)}{2(2-\lambda)}$, $|a_3| \le \frac{\xi''(0)}{4(3-2\lambda)}$ and

$|a_4| \le \frac{1}{4(4-3\lambda)}\left(\frac{\xi''(0)\xi'(0)}{2(3-2\lambda))} - \frac{(2+\lambda)\xi''(0)\xi'(0)}{8(2-\lambda)(3-2\lambda)} + \frac{\xi'''(0)}{6}\right).$

The next corollary is a result from Theorem2.1 and Lemma 1.1.

Corollary 2.1 .Let $f \in SK_s(\lambda, A, B)$ for $-1 \le B < A \le 1$. Then

$$|a_2| \le \frac{(A-B)}{2(2-\lambda)}$$

$$|a_3| \le \frac{(A-B)}{4(3-2\lambda)}$$

$$|a_4| \le \frac{1}{4(4-3\lambda)}\left(\frac{(A-B)^2}{(3-2\lambda))} - \frac{(2+\lambda)(A-B)^2}{4(2-\lambda)(3-2\lambda)} + (A-B)\right).$$

In this paper, the Fekete-Szego inequality for a univalent normalized functions f in the subclass $SK_s(\lambda, A, B)$ is obtained .

Theorem2.2. Let $\xi(z) = 1 + \xi_1 z + \xi_2 z^2 + \cdots$ and $f \in SK_s(\lambda, \xi)$. Then

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{\xi''(0)}{2(3-2\lambda)} - \mu \frac{(\xi'(0))^2}{(2-\lambda)^2} & if \ \mu \le \delta_1 \\ \frac{\xi''(0)}{2(3-2\lambda)} & if \ \delta_1 \le \mu \le \delta_2 \\ \frac{-\xi''(0)}{2(3-2\lambda)} + \mu \frac{(\xi'(0))^2}{(2-\lambda)^2} & if \ \mu \ge \delta_2 \end{cases}.$$

Where

$$\delta_1 = \frac{(\xi_1 - \xi_2)(2-\lambda)}{\xi_1^2(3-2\lambda)}, \quad \delta_2 = \frac{(\xi_1 + \xi_2)(2-\lambda)}{\xi_1^2(3-2\lambda)}$$

And

$$\mu = \frac{4(2-\lambda)(\xi_1 - \xi_2) - \xi_1^2((3-2\lambda))}{\xi_1^2(3-2\lambda)}.$$

Proof. If $p(z) = \lambda \frac{zf'(z)}{\varphi_0(f(z))} + (1-\lambda)\frac{(zf'(z))'}{\varphi_1(f'(z))} = 1 + p_1 z + p_2 z^2 + \cdots$

then $2(2-\lambda)a_2 = p_1$ and $2(3-2\lambda)a_3 = p_2$.

Let $p \prec \xi$ such that $p^{-1}(z) = \frac{1+\xi^{-1}(p(z))}{1-\xi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots (= p_1(z))$ is an analytic function with a positive real part in $\mathcal{D}$. Then

$$p(z) = \xi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = \xi\left(\frac{c_1 z + c_2 z^2 + \cdots}{2 + c_1 z + c_2 z^2 + \cdots}\right) = \xi\left(\tfrac{1}{2} c_1 z + \tfrac{1}{2}(c_2 - \tfrac{1}{2}c_1^2)z^2 + \cdots\right)$$

$$= \tfrac{1}{2}\xi_1 c_1 z + \tfrac{1}{2}\xi_1(c_2 - \tfrac{1}{2}c_1^2)z^2 + \tfrac{1}{4}\xi_2 c_1^2 z^2 + \cdots = 1 + p_1 z + p_2 z^2 + \cdots$$

Thus $p_1 = \frac{1}{2}\xi_1 c_1$, $p_2 = \frac{1}{2}\xi_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}\xi_2 c_1^2$

And $a_2 = \frac{\xi_1 c_1}{4(2-\lambda)}$, $a_3 \leq \frac{\xi_1(c_2-\frac{1}{2}c_1^2)+\frac{1}{2}\xi_2 c_1^2}{4(3-2\lambda)}$.

So that

$$a_3 - \mu a_2^2 = \frac{\xi_1}{4(3-2\lambda)}(c_2 - \tfrac{1}{2}c_1^2) + \frac{\xi_2 c_1^2}{8(3-2\lambda)} - \mu \frac{\xi_1^2 c_1^2}{16(2-\lambda)^2}$$

$$= \frac{\xi_1}{4(3-2\lambda)}\left(c_2 - c_1^2\left(\frac{1}{2}\left(1 - \frac{\xi_2}{\xi_1} - \mu\frac{\xi_1(3-2\lambda)}{2(2-\lambda)^2}\right)\right)\right).$$

Therefore

$$a_3 - \mu a_2^2 = \frac{\xi'(0)}{4(3-2\lambda)}(c_2 - c_1^2 \nu)$$

where$\nu = \frac{1}{2}\left(1 - \frac{\xi''(0)}{2\xi'(0)} - \mu\frac{\xi'(0)(3-2\lambda)}{2(2-\lambda)^2}\right)$.

Assume that

$$\delta_1 = \frac{(2\xi'(0)-\xi''(0))(2-\lambda)}{(\xi'(0))^2(3-2\lambda)} \quad \text{and} \quad \delta_2 = \frac{(2(\xi'(0)+\xi''(0))(2-\lambda)}{(\xi'(0))^2(3-2\lambda)}$$

So that, If $\mu \leq \delta_1$ then

$$|a_3 - \mu a_2^2| \leq \frac{\xi''(0)}{2(3-2\lambda)} - \mu\frac{(\xi'(0))^2}{(2-\lambda)^2}$$

and if $\delta_1 \leq \mu \leq \delta_2$ then

$$|a_3 - \mu a_2^2| \leq \frac{\xi''(0)}{2(3-2\lambda)}$$

also, if $\mu \geq \delta_2$ then

$$|a_3 - \mu a_2^2| \leq -\frac{\xi''(0)}{2(3-2\lambda)} + \mu\frac{(\xi'(0))^2}{(2-\lambda)^2}$$

To discuss the equality in the above bounds, we must define the following operators:

(I) $\Upsilon_\phi = \frac{z K'_{\phi_n}(z)}{\varphi_0(K_{\phi_n}(z))} = \phi(z^{n-1}),\quad K'_{\phi_n}(0) - 1 = 0 = K_{\phi_n}(0).$

(II) $\Upsilon_\lambda = \frac{z J'_\lambda(z)}{\varphi_0(J_\lambda(z))} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right),\ J'_\lambda(0) - 1 = 0 = J_\lambda(0),\quad for\ (0 \leq \lambda \leq 1).$

(III) $\Upsilon_{-\lambda} = \frac{z H'_\lambda(z)}{\varphi_0(H_\lambda(z))} = \phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right),\ H'_\lambda(0) - 1 = 0 = H_\lambda(0),\ for\ (0 \leq \lambda \leq 1)$

where the operators Y_ϕ, Y_λ and $Y_{-\lambda}$ are in $SK_s(\lambda, A, B)$.

By Lemma 1.5, if $\mu = \delta_1$ then the equality holds if and only if $J_\lambda = f$ or one of its rotations, if $\mu = \delta_2$ then the equality holds if and only if $H_\lambda = f$ or one of its rotations. If $\delta_1 < \mu < \delta_2$ then the equality holds if and only if $K_{\phi_3} = f$ or one of its rotations. And if $\mu > \delta_2$ or $\mu < \delta_1$ then the equality holds if and only if $K_{\phi_2}(z) = f$ or one of its rotations.□

Corollary 2.2. Let $f \in SK_s(\lambda, A, B)$ for $-1 \le B < A \le 1$.Then

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{(A-B)}{2(3-2\lambda)} - \mu\dfrac{(A-B)^2}{(2-\lambda)^2} & if\ \ \mu \le \delta_1 \\ \dfrac{(A-B)}{2(3-2\lambda)} & if\ \delta_1 \le \mu \le \delta_2 \\ \dfrac{-(A-B)}{2(3-2\lambda)} + \mu\dfrac{(A-B)^2}{(2-\lambda)^2} & if\ \ \mu \ge \delta_2 \end{cases} .$$

The proof is obvious from Lemma 1.1 and Theorem 2.2.

The next result is needed for Hankal determinant inequalities for a function f in $SK_s(1, A, B)$ from Theorem 2.1 and Lemma 1.2 when $\lambda = 1$.

Corollary 2.2.Let $f \in SK_s(1, A, B)$for $-1 \le B < A \le 1$. Then

$$|a_2 a_4 - a_3^2| \le \frac{1}{24} \quad and \quad |a_2 a_4 - a_3^2| \le \frac{7}{14}.$$

Corollary 2.2.Let $f \in SK_s(\lambda, A, B)$for $-1 \le B < A \le 1$ for $0 \le \lambda \le 1$. Then the differential equation

$$\lambda\frac{\partial f(z)}{\partial z}\,\varphi_1\left(\frac{\partial f(z)}{\partial z}\right) + (1-\lambda)\frac{\partial(zf'(z))}{\partial z}\,\varphi_0\big(f(z)\big) = 0$$

has the solution

$$f(z) = \beta\int \varphi_0\big(f(z)\big)\,\partial z,$$

$$for\ \ some\ \ \beta \in [1, \infty).$$

Proof. Since

$$\lambda\frac{\varphi_1\left(\frac{\partial f(z)}{\partial z}\right)}{\varphi_0\big(f(z)\big)} = (\lambda - 1)\frac{\frac{\partial(zf'(z))}{\partial z}}{\frac{\partial f(z)}{\partial z}}$$

Then

$$\lambda \ln\left(\varphi_0(f(z))\right) = (\lambda - 1)\ln\left(\frac{\partial f(z)}{\partial z}\right) + C,$$

Without any loss of generality, let $C = 0$and $\beta = \exp\left(\frac{\lambda}{\lambda-1}\right)$. Then

$$f(z) = \beta \int \varphi_0(f(z))\,\partial z \qquad \text{for } 1 \leq \beta \leq \infty.\square$$

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